

ALGEBRAS WITH A NEGATION MAP

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ABSTRACT. Our objective in this is three-fold, the first two covered in this paper. In tropical mathematics, as well as other mathematical theories involving semirings, when trying to formulate the tropical versions of classical algebraic concepts for which the negative is a crucial ingredient, such as determinants, Grassmann algebras, Lie algebras, Lie superalgebras, and Poisson algebras, one often is challenged by the lack of negation. Following an idea originating in work of Gaubert and the Max-Plus group and brought to fruition by Akian, Gaubert, and Guterman, we study algebraic structures with negation maps, called **systems**, in the context of universal algebra, showing how these encompass the more viable (super)tropical versions, as well as hypergroup theory. Special attention is paid to **meta-tangible** systems, whose algebraic theory is rich enough to provide a host of structural results. Basic results also are obtained in linear algebra, linking determinants to linear independence.

Formulating the structure categorically enables us to view the tropicalization functor as a morphism, thereby further explaining the mysterious link between classical algebraic results and their tropical analogs, as well as with hyperfields. We use the tropicalization functor to analyze some tropical structures and propose tropical analogs of classical algebraic notions.

In work currently in progress, having the basic category in place, we proceed to the third stage, a theory of sheaves and schemes and derived categories with negation.

CONTENTS

1. Introduction	3
1.1. A broad overview	4
1.2. Preliminaries	5
1.3. Introducing negation maps	7
1.4. The main tropical examples	8
1.5. Optional: Hypergroups	12
1.6. Introducing surpassing relations	13
1.7. Introducing systems	15
1.8. Introducing symmetrization	17
2. Background from universal algebra	19
2.1. Algebraic structures	19
2.2. Varieties	20
2.3. Partial orders in universal algebra	21
2.4. (Optional) Application: \mathcal{T} -semirings [†] and their modules.	22
2.5. Congruences	23
2.6. Free algebras	23

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2.7. Homogeneous and multilinear operators	24
3. Tropical examples viewed in terms of universal algebra	25
3.1. Varieties arising naturally in tropical mathematics	25
3.2. Structures of tropical mathematics which do not comprise varieties	26
4. Negation maps and surpassing relation	27
4.1. Negation maps in universal algebra	27
4.2. Combining and comparing negation maps	28
5. Systems	29
5.1. A general overview of systems	29
5.2. Uniquely negated triples	29
5.3. The characteristic of a triple	31
5.4. Neutral elements	31
5.5. Polynomials and their roots	31
5.6. Triples with involution	32
5.7. Hyperstructures	33
6. Meta-tangible triples and their systems	34
6.1. The characteristic of a meta-tangible triple	36
6.2. Cancellative meta-tangible triples	37
6.3. Uniform elements and height in meta-tangible triples	38
6.4. Surpassing relations on meta-tangible triples	40
6.5. Most meta-tangible systems are matroidal and \mathcal{T} -reversible	41
6.6. \mathcal{T} -classical meta-tangible triples	43
6.7. Squares and sums of squares	44
6.8. Sign maps on \mathcal{T}	45
6.9. Classifying metasystems	46
6.10. Important examples of meta-tangible systems	46
6.11. Meta-tangible systems versus meta-tangible hypergroups	48
7. Symmetrization	49
7.1. Supermodules and super-semialgebras	49
7.2. Symmetrization of modules	49
7.3. Symmetrization of semirings and \mathcal{T} -semirings	50
7.4. Symmetrization in the language of universal algebra	50
7.5. Modification of symmetrization	51
7.6. The transfer principle	52
8. Categories of systems	52
8.1. Categories with negation	52
8.2. Morphisms of systems	52
8.3. Embedding hypergroups into systems	53
8.4. Congruences and \mathcal{T} -ideals on systems	53
8.5. Tensor products with a negation map, and their semialgebras	55
9. Linear algebra over a uniquely negated triple	56
9.1. Matrices over systems	56
9.2. Dependence relations of vectors	58
9.3. Ranks of matrices	59
10. Tropicalization	60
10.1. Tropicalization of Puiseux series	60
10.2. Tropicalization of classical systems defined over Puiseux series	61
11. Tropical structures arising from tropicalization	61
11.1. Exterior (Grassmann) semialgebras with a negation map	61
11.2. Nonassociative algebras with a negation map	63
11.3. Lie semialgebras and Lie super-semialgebras	63
11.4. Poisson algebras and their module congruences	64
12. Appendix A: Hyperfields as systems	65
12.1. Power sets of semigroups	65

12.2. Major examples of hypergroups and hyperfields	67
13. Appendix B: Fuzzy rings	69
References	70

1. INTRODUCTION

This paper was motivated by the desire to understand a mysterious parallel between structural results in what we will call the “classical algebraic theory” and theorems formulated directly in varied aspects of tropical algebra, despite the former being taken over fields and the latter over the max-plus algebra and related semifields. It is designed to lay the foundation for a unified algebraic theory, which also encompasses diverse recent research, especially in the tropical setting and for hyperfields, and to a lesser extent, “fuzzy rings.” But our initial point of view is tropical. Once the overall framework, the **uniquely negated system**, is established, it provides a mechanism for obtaining effective definitions of new tropical algebraic structures, and also provides a guide for applying classical algebraic techniques in these other situations. For example, a useful definition of “characteristic” is given in Definition 5.10.

Tropicalization originally was viewed as a limiting process taking logarithms and passing in the limiting case to the max-plus algebra, which is a semiring. Thus, tropical algebra always has relied on the theory of semirings, which goes back to Costa [17] and Eilshauer [25], and for which we use [30] as our standard reference. But lack of negatives obviously hampers the algebraic theory.

Over the years, various researchers, going back to Kuntzman [57] in 1972, have tackled the lack of negation in the max-plus semiring, especially for matrices and the determinant. Some have used an operation resembling negation. Gaubert [26] introduced such a structural approach in his dissertation, motivated by [65, 71]; see for example [65, p.352, end of proof of (a)]. His work has been continued together with the M. Plus group and Akian and Guterman and Henry, using a “symmetrized” theory ([64], [26], [6, §3.4], [27], [2], [35], [3], and [51, Appendix A]) leading to a general “transfer principle” to generate semiring identities. More recently, Bertram and Easton [8] and Joo and Mincheva [52] have utilized the “twist” of [2] to refine congruences on polynomial semirings.

As the field of Puiseux series came into play, the underlying semiring was viewed as the target of the Puiseux valuation, which differs somewhat from the max-plus algebra. Towards this end, in [38, 41, 46, 49] a “supertropical” theory was initiated over a semiring R by means of a “ghost map” (where the negation map actually is the identity map), with various applications to affine varieties, matrices, linear algebra, and quadratic forms. In [2] and [47, 49] it was possible to transfer classical algebraic results to the tropical theory by means of a somewhat mysterious “surpassing” relation on semirings, which satisfies many properties of equality, and replaces equality in many generalizations of classical theorems, especially for polynomials and matrix theory. This is given in Example 1.41(ii) for supertropical algebra, and in Example 1.41(iii) for Gaubert’s diodes. Thus we are motivated to ask exactly how this surpassing relation fits into the algebraic theory.

The same kind of relation (this time, as a subset), has also turned up in the theory of hypergroups. Viro [74] has viewed tropical theory in terms of hyperfields, and it turns out that the hyperfield theory can be embedded into the theory, as spelled out in Appendix A. Indeed, the recent spur in research in hypergroups provides grounds for the study of “uniquely negated systems.” Recently ties have been found in [29] between hyperfields and fuzzy rings, and these also are reflected in the theory.

In all of these instances, there is a set \mathcal{T} of main interest (the max-plus algebra, the symmetrized max-plus algebra, and the underlying hypergroup), together with a special operation resembling negation (the switch map or the hypernegative in the latter two cases), but its intrinsic algebraic structure is not sufficient for satisfactory investigation, causing many algebraic results to be formulated and proved on an ad hoc basis. The situation is significantly clarified by embedding \mathcal{T} into a much richer structure \mathcal{A} (such as supertropical, symmetrized, or power set) which can be studied via well-known techniques from universal algebra. The interplay between \mathcal{A} and \mathcal{T} is intriguing. In brief, a “system” is comprised of an algebraic structure \mathcal{A} , (often a semiring), a designated subset \mathcal{T} , a **negation map** $(-)$, and the **surpassing relation** \preceq .

Our major goal with these “systems” is to build an algebraic foundation that unifies all of these approaches in a way that also includes the classical algebraic theory, and in which the “surpassing

relation” is an intrinsic component that encompasses equality. But we want the axioms to be sufficiently restrictive to specialize naturally to our main examples from tropical mathematics and the theory of hypergroups, thereby providing an axiomatic set-up that will drive the theory, showing the way to natural new definitions, and eventually yielding intrinsic theorems.

The appropriate setting for the investigation seems to be that of universal algebra, to be reviewed below in 2.5, where we start with addition as the basic operation (of a semigroup), treated differently from all others, and bring in other operations as seen fit. This provides a vehicle (the “system”) for linking more sophisticated theorems from classical algebra and algebraic geometry to tropical algebra (and also to hyperfields). This can be viewed in context of Lorscheid’s “blueprint,” but also involves specific extra information to permit us to hone in on the applications, as described in Sections 4,5,6,7.

Systems behave well categorically, especially under tensor products, as considered in §8. In particular one can turn to modules and linear algebra (§9). They meld well with tropicalization (§10), inspiring tropical versions of classical algebraic structures in §11.

Since the tropical structures are distributive, but the applications to hyperfields and fuzzy rings are not necessarily distributive, and sometimes rather technical, we defer some details to Appendices A and B respectively. The applications to hyperfields are direct and yield immediate results in hyperfield theory, via the point of view of §12.2.1. The applications to fuzzy rings are more tenuous, but in fact give rise to a slightly more general version (Definition 13.2) which “explains” the role of invertible elements through the structural results in Proposition 13.7 and Proposition 13.8.

1.1. A broad overview.

We start with some semigroup $(\mathcal{A}, +)$, perhaps with extra structure (often multiplication), in which we are interested in a certain subset \mathcal{T} called the **tangible elements**. \mathcal{T} could be all of \mathcal{A} in classical algebra, or an ordered subgroup identified with the max-plus algebra (or related structures) in tropical algebra, or a hyperfield. Usually, \mathcal{A} is taken to be a semiring. However, to accommodate application to hyperrings we might relax distributivity, cf. §2.4.

Inspired by [2], we couple addition with a **negation map** (Definition 1.9), which is a formal map $a \mapsto (-)a$ that satisfies all of the properties of negation *except* $a + ((-)a) = 0$. This comes automatically for classical algebra and for hyperfields. Initially, negation is notably absent in the tropical theory, but is circumvented in two main ways: The identity itself is a negation map, leading to the “supertropical theory,” or else one can introduce a negation map, called a “symmetry” in [2], through the process of “symmetrization” (§1.8), passing to $\mathcal{A} \times \mathcal{A}$.

To simplify notation, we write $a(-)b$ for $a + ((-)b)$. To avoid ambiguity, we then write the product of a and $(-)b$ as $a((-)b)$, which occurs much more rarely. Also we write $(\pm)a$ for “ a or $(-)a$,” and $a(\pm)b$ for “ $a + b$ or $a(-)b$.”

As to be expected, the flavor of the theory differs according to whether or not $(-)$ is the identity map, called respectively the **first** and **second** kind (Definition 1.10). This enables us to distinguish between “supertropical” and “symmetrized” tropical algebra, and helps to explain why theorems for supertropical algebras might fail for symmetrized algebras, as illustrated in [5].

We define $a^\circ := a(-)a$, called a **quasi-zero**. When considering structures with a multiplicative unit element 1 (defined as satisfying $1a = a = a1$, $\forall a$), we define $e := 1^\circ = 1(-)1$.

So far we have the data $(\mathcal{A}, \mathcal{T}, (-))$, which we call a **triple**. We say that the triple is **uniquely negated** when for any a in \mathcal{T} , $(-)a$ is the only element in \mathcal{T} for which $a(-)a$ is a quasi-zero. This already is a rather powerful condition, with some crucial consequences given in Proposition 5.2 and Corollary 5.3.

Next we define a **surpassing relation** \preceq , the major example being \preceq_\circ defined by $a \preceq_\circ b$ iff $b = a + c^\circ$ for some $c \in \mathcal{A}$. We write $a \succeq b$ when $b \preceq a$. In order to obtain that \preceq_\circ is a surpassing relation requires a basic assumption on the triple $(\mathcal{A}, \mathcal{T}, (-))$, such as **meta-tangibility** (Definition 1.51), characterized by the property that the sum of tangible elements that are not quasi-negatives of each other is tangible. (Another example of surpassing relation is \subseteq arising in the theory of hypergroups.) Ironically, instead of being symmetric (and thus an equivalence), in the non-classical cases the surpassing relation often is antisymmetric, although it restricts to equality on \mathcal{T} .

Altogether, our structure of choice, a **system** (Definition 5.1), is a quadruple $(\mathcal{A}, \mathcal{T}, (-), \preceq)$, where \mathcal{T} is the set of tangible elements, generating \mathcal{A} additively, $(-)$ is a negation map, and \preceq is a surpassing relation. This covers the classical case, the “standard” supertropical semiring, the “symmetrized” semiring of [2, 3],

the “exploded” algebra [62], the “layered” semiring of [40], and some hyperfields (but not all, cf. the examples in [7], reviewed in Remark 12.4).

Expressed in these terms, one of the obstacles to tropical structure theories has been to describe a° accurately. The answer seems to be to treat this differently from $a+b$ where $b \neq (-)a$, such as in uniquely negated systems (Definition 1.50). Another example — we define $(-)$ -**bipotence** by $a+b \in \{a, b\}$ when $b \neq (-)a$ (Definition 1.51); this turns out to classify precisely the variants of the max-plus algebra that have arisen in the tropical literature.

In §5.5 we discuss polynomials and their roots, to pave the way for affine geometry (but not in this paper).

There are two ways of approaching systems — one is as the basic algebraic structure, such as a ring, and the other is as a secondary structure (such as a module or hyper-module) which provides insight into the former. Our emphasis in this paper is on the former, since one has to pause somewhere, and 72 pages seems enough. This covers the basic tropical algebraic structures, hypergroups, and analogs of classical constructions.

Meta-tangible triples and their systems, lying at the center of this study, are treated in considerable detail in §6, where we show in Theorem 6.18 that the cancellative meta-tangible triples are either $(-)$ -bipotent (Definition 1.51) or satisfy $e + \mathbb{1} = \mathbb{1}$. Their elements all have the “uniform” presentation given in Theorem 6.25. Cancellative meta-tangible systems are classified in Theorem 6.57, often reducing to the familiar examples from tropical theory.

One important application of systems (which actually motivated this paper) is the symmetrization process, studied in these terms in §7.

Seeing that uniquely negated systems, especially meta-tangible systems, have a robust algebraic theory, we proceed to view them categorically in §8, utilizing the surpassing relation \preceq as an essential ingredient in the definition of morphism in Definition 8.4. Tensor products are defined in this context, although one could lose meta-tangibility. This approach leads to an intrinsic version of tropicalization, as a morphism of systems, given in §10.

Linear algebra over systems is particularly intriguing, since some of the supertropical results go over, but others have counterexamples, as discussed in §9. The differences largely come from whether negation is of the first or second kind. This delicate issue is treated explicitly in [5].

For those researchers also interested in hypergroups, we include a semiring-like structure (\mathcal{T} -semirings) from universal algebra, satisfying distributivity only over \mathcal{T} , encompassing hypergroups and hyperfields, cf. §2.4. This specialized notion actually helps our intuition, since its assortment of examples, given in Appendix A, casts a strong light on the axiomatic theory.

1.2. Preliminaries.

In order to elaborate further the results in this paper, we need some preliminary terminology and facts. As customary, \mathbb{N} denotes the positive natural numbers, \mathbb{N}_0 denotes $\mathbb{N} \cup \{0\}$, \mathbb{Q} the rational numbers, and \mathbb{R} the real numbers, all ordered monoids under addition.

1.2.1. Ongoing hypothesis.

From now on, we carry the ongoing hypothesis that $(\mathcal{A}, +)$ is a semigroup, with a distinguished subset \mathcal{T} of \mathcal{A} that additively generates \mathcal{A} . When \mathcal{A} contains an element 0 , we write \mathcal{T}_0 for $\mathcal{T} \cup \{0\}$ and assume that \mathcal{T}_0 additively generates \mathcal{A} . (Presuming that the extra operators preserve \mathcal{T} in the appropriate sense, this situation is realized when one replaces \mathcal{A} by the additive sub-semigroup spanned by \mathcal{T}_0 , so this requirement is rather mild.)

This hypothesis enables us to define multilinear operators (§2.7) on \mathcal{A} via their action on the elements of \mathcal{T} , and to lift various properties from \mathcal{T} to \mathcal{A} . We define the **height** of an element $c \in \mathcal{A}$ as the minimal t such that $c = \sum_{i=1}^t a_i$ with each $a_i \in \mathcal{T}$. (We say that 0 has height 0.) The **height** of \mathcal{A} is the maximal height of its elements. Thus \mathcal{A} has height 1 iff $\mathcal{A} = \mathcal{T}$ or \mathcal{T}_0 , and height 2 also will play an important role. Height 3 involves extra considerations, as indicated for example in Definition 6.24 and Theorem 6.25.

1.2.2. General preliminaries.

Recall that a **monoid** is a semigroup with a two-sided identity element, denoted as 0 for addition, and $\mathbb{1}$ for multiplication. For any multiplicative semigroup $\mathcal{M} := (\mathcal{M}, \cdot)$ we can formally adjoin the identity

element $\mathbb{1}_M$ by declaring that $\mathbb{1}_M \cdot a = a \cdot \mathbb{1}_M = a$ for all $a \in M$, and when dealing with multiplication we always work with monoids. We customarily write ab for $a \cdot b$.

In additive notation, when $(M, +)$ is a semigroup, we write M_{+0} for the monoid $M \cup \{0\}$ where $\{0\}$ is formally adjoined satisfying $0 + a = a + 0 = a$ for all $a \in M$.

A **semiring**[†] is a semiring $(R, +, \cdot, \mathbb{1}_R)$ without 0 , i.e., an additive Abelian semigroup $(R, +)$ and multiplicative monoid $(R, \cdot, \mathbb{1}_R)$ satisfying the usual distributive laws. The reason that we do not always require R necessarily to have the element 0 is that 0 just distracts from our true goal, which is negation maps and quasi-zeros, not negatives.¹ One may assume that R has an undesignated element 0 if one so wishes. The theory of semirings[†] is essentially the same as that of semirings.

Definition 1.1. A semiring[†] $(R, +, \cdot, \mathbb{1}_R)$ is a **semifield**[†] if (R, \cdot) is an Abelian group. A semiring[†] R is **idempotent** if $a + a = a$ for all $a \in R$, R is **bipotent** if $a + b \in \{a, b\}$ for all a, b .

The max-plus algebra is bipotent, but bipotence (barely) fails in the other examples, thereby motivating $(-)$ -bipotence (Definition 1.51).

Definition 1.2. An R -**module** (often called a **semimodule** in the literature) over a semiring[†] R is an additive monoid $(M, +, 0_M)$ together with scalar multiplication $R \times M \rightarrow M$ paralleling the module axioms of classical algebra, although now one must stipulate that $r0_M = 0_M$ for all r in R .²

For example, $R_{+0} := R \cup \{0_R\}$ is naturally an R -module. Likewise, one defines submodules and homomorphic images, and the semigroup $(\text{Hom}(M, N), +, 0)$, for any R -modules M and N . We say that a module can be written in the form $M_1 \oplus M_2$ when every element can be written uniquely as the sum of an element of M_1 and M_2 . Then we can define direct sums, and $M^{(I)} := \bigoplus_{i \in I} M_i$ where each $M_i = M$.

Definition 1.3. The **free module** over a semiring[†] R is $R_{+0}^{(I)}$, denoted $R^{(I)}$ for short, where we identify the base element e_i with $\mathbb{1}_R$ in the i component.

We may consider semiring modules rather than the underlying semirings, as in classical representation theory. Many concepts do not involve module multiplication, and are formulated for additive semigroups. In the other direction the following simple observation enables us to apply module theory to semigroups:

Remark 1.4. Any semigroup is an \mathbb{N} -module in the obvious way.

Definition 1.5. As in classical algebra, a **semialgebra** over a commutative (associative) semiring[†] C is a module \mathcal{A} which also has a multiplication with respect to which it becomes a semiring satisfying the usual law

$$c(a_1 a_2) = a_1 (c a_2) = (c a_1) a_2, \quad \forall c \in C, a_i \in \mathcal{A}. \quad (1.1)$$

At the outset, we work with associative semialgebras but later on they can be nonassociative, according to the context.

We recall the usual definition of the **monoid semialgebra** $C[M]$ of a monoid M over a commutative, associative semiring[†] C , by taking the free module over C whose base is the elements of M , with multiplication induced by the given multiplication in C and in M , extended via distributivity.

Definition 1.6. A **partial pre-order** is a transitive relation satisfying $a \leq a$ for all a .

A **partially pre-ordered monoid** is a monoid M with a partial pre-order satisfying

$$a \leq b \quad \text{implies} \quad ca \leq cb, \quad ac \leq bc \quad (1.2)$$

for all elements $a, b, c \in M$. A monoid M is **ordered** (resp. **partially ordered**) if its pre-order (resp. partial pre-order) is antisymmetric. We write PO for partial order.

Remark 1.7. $(\mathbb{Q}^\times, \cdot)$ is not an ordered monoid under this definition, since taking inverses reverses the order.

¹To be fair, one often wants a zero element in order to be able to define such familiar algebraic varieties as $xy = 0$, but also this could be viewed as the asymptote of the hyperbolas $xy = c$ as c decreases.

²If instead we study modules over semirings with zero 0_R then we also stipulate that $0_R a = 0_M, \forall a \in M$. This leads to ambiguity in defining modules M over semirings[†] containing a zero element 0_R that has not been designated as such; to resolve this ambiguity, one could mod M out by the equivalence given by $0_R a_1 \equiv 0_R a_2$ for all $a_i \in M$.

Recall that a **(nonarchimedean) valuation** from a ring R to an ordered monoid $(\mathcal{G}, +, 0)$ is a monoid homomorphism $v : (R, \cdot) \rightarrow \mathcal{G}$ satisfying

$$v(a + b) \geq \min\{v(a), v(b)\}, \quad \forall a, b \in R.$$

It is well known that $v(\pm 1) = 0$, and if $v(a) > v(b)$ then $v(a + b) = v(b)$.

1.2.3. Digression: Modules over monoids.

For hypergroups, as we shall see shortly, addition passes outside the original set, which is why the following more general version of modules could come in handy.

Definition 1.8. A monoid $(\mathcal{T}, \cdot, \mathbb{1})$ **acts** on a set \mathcal{S} if there is a multiplication $\mathcal{T} \times \mathcal{S} \rightarrow \mathcal{S}$ satisfying $\mathbb{1}s = s$ and $(a_1 a_2)s = a_1(a_2 s)$ for all $a_i \in \mathcal{T}$ and $s \in \mathcal{S}$.

A **module** \mathcal{A} over a monoid $(\mathcal{T}, \cdot, \mathbb{1})$ is an Abelian semigroup $(\mathcal{A}, +, \mathbb{0}_{\mathcal{A}})$ on which \mathcal{T} acts, also satisfying the condition:

$$a\mathbb{0}_{\mathcal{A}} = \mathbb{0}_{\mathcal{A}}, \quad \forall a \in \mathcal{T}.$$

1.3. Introducing negation maps.

Definition 1.9. A **negation map** on an additive semigroup $(\mathcal{A}, +)$ is a semigroup homomorphism $(-) : \mathcal{A} \rightarrow \mathcal{A}$ of order ≤ 2 , written $a \mapsto (-)a$. (Thus $(-)(a + b) = (-)a + (-)b$.)

A **negation map** on a module M over a semiring[†] $(R, \cdot, +, \mathbb{1}_R)$ is simultaneously a negation map on the additive semigroup $M(+)$, as well as satisfying

$$(-)(ra) = r((-)a), \quad \forall r \in R, a \in M. \quad (1.3)$$

A **negation map** on a semiring[†] $(R, \cdot, +, \mathbb{1}_R)$ is simultaneously a negation map on the additive semigroup $(R, +)$, as well as satisfying

$$(-)(a_1 a_2) = ((-)a_1)a_2 = a_1((-)a_2), \quad \forall a_i \in R. \quad (1.4)$$

Negation maps are best understood as 1-ary operators in universal algebra³, as developed in §4, but here is the specific case of main interest.

The two obvious examples are the identity map, and (for modules over rings) the usual negation map $(-)a = -a$. This gives rise to two kinds of negation maps.

Definition 1.10. A negation map $(-)$ is of the **first kind** if $(-)a = a$ for all $a \in \mathcal{A}$. The negation map is of the **second kind** if $(-)a \neq a$ for some $a \in \mathcal{T}$.

Recall that $(\mathcal{A}, \mathcal{T}(\mathcal{A}), (-))$ is called a **triple**.

Definition 1.11. A triple $(\mathcal{A}, \mathcal{T}(\mathcal{A}), (-))$ is **\mathcal{T} -cancellative** over a monoid $(\mathcal{T}, \cdot, \mathbb{1})$ acting on \mathcal{A} , if $a_1 b = a_2 b$ implies $a_1 = a_2$ for $a_1, a_2 \in \mathcal{T}$ and $b \in \mathcal{T}(\mathcal{A})$. A \mathcal{T} -cancellative triple is **\mathcal{T} -invertible** if \mathcal{T} is a multiplicative group.

In this paper, we always have $\mathcal{T} = \mathcal{T}(\mathcal{A})$, with (\mathcal{T}, \cdot) acting on \mathcal{A} in the natural way. In this context, we write **cancellative** for “ \mathcal{T} -cancellative.”

Lemma 1.12. When the triple $(\mathcal{A}, \mathcal{T}, (-))$ is cancellative, the negation map $(-)$ is of the first kind iff $(-)\mathbb{1} = \mathbb{1}$; $(-)$ is of the second kind iff $(-)\mathbb{1} \neq \mathbb{1}$.

Proof. $(-)\mathbb{1} = \mathbb{1}$ iff $(-)a = a$ for all a , since we can cancel a . □

As in [3], one has:

Definition 1.13. When R already has a negation map, we say that the negation maps on R and M are **compatible** if

$$(-)(ra) = ((-)r)a = r((-)a), \quad \forall r \in R, a \in M.$$

³It is convenient to cast our considerations in terms of universal algebra. Although more complicated than the usual algebraic structure theory because an intrinsic negative is not available, universal algebra enhances the tropical and supertropical structures, and has a wide range of applications given in §3.1.

Example 1.14. Suppose R already has a negation map $(-)$, with $\mathbb{1} \in R$. Then any R -module M has a compatible negation map given by $(-)a = ((-)\mathbb{1}_R)a$. Thus, we can view the negation map on M in terms of the single element $(-)\mathbb{1}$.

Also, any module homomorphism φ satisfies $\varphi((-)a) = \varphi((-)\mathbb{1}_R a) = ((-)\mathbb{1}_R)\varphi(a) = (-)\varphi(a)$.

This raises the question of how to cope simultaneously with different negation maps at once, which we discuss briefly in §4.2. But our applications are for a single given negation map on M .

The following notion takes the role customarily assigned to the zero element. Recall that a° denotes $a(-)a$.

Remark 1.15. $a^\circ = ((-))a^\circ$.

Lemma 1.16. If $\emptyset \in (\mathcal{A}, +)$, then $(-)\emptyset = \emptyset$.

Proof. $(-)\emptyset = (-)\emptyset + \emptyset = (-)\emptyset + ((-)(-)\emptyset) = (-)(\emptyset(-)\emptyset) = (-)((-)\emptyset) = \emptyset$. \square

Definition 1.17. Given a semigroup $(\mathcal{A}, +)$ and a subset $\mathcal{T} \subseteq \mathcal{A}$, we denote

$$\mathcal{T}^\circ = \{a^\circ : a \in \mathcal{T}\}, \quad \mathcal{T}^+ := \mathcal{T}_0 \cup \mathcal{T}^\circ.$$

\mathcal{T}° is the analog of the “balanced elements” of [2].

Sometimes we write M instead of \mathcal{A} when we want to stress the module structure.

Lemma 1.18 ([2, Remark 4.5]). M° is a submodule of M , for any module M with negation map.

Proof. $\emptyset^\circ = \emptyset(-)\emptyset = \emptyset + \emptyset = \emptyset \in M^\circ$, and $r(a^\circ) = r(a(-)a) = (ra)(-)(ra) = (ra)^\circ$. \square

The introduction of the negation map to replace negatives enables us to develop the tropical analogs of some of the most basic structures of algebra, applicable to Parker’s exploded algebra [62], Sheiner’s ELT algebra [70], Grassmann algebras [28], Blachar’s ELT Lie algebras [11], Lie super-semialgebras, and Poisson algebras, and unifies research coming from different directions as well.

Occasionally we want the following notion.

Definition 1.19. We say that a module M is \circ -ordered if M° is ordered, and we write $a_1 >_\circ a_2$ if $a_1^\circ > a_2^\circ$.

In classical algebra, the only \circ -order is trivial, since $a^\circ = \emptyset$ for all a .

Definition 1.20. Often we are given a multiplication $\mathcal{T} \times \mathcal{A} \rightarrow \mathcal{A}$. We say that \mathcal{A} **contains** $\mathbb{1}$ if there is an element $\mathbb{1} \in \mathcal{T}$ such that $\mathbb{1}a = a$ for all $a \in \mathcal{A}$. For example, this holds in any semiring[†] with a negation map, and we designate several important elements, for future reference:

$$e = \mathbb{1}(-)\mathbb{1}, \quad e' = e + \mathbb{1}, \quad e^\circ = e + e. \quad (1.5)$$

Also we define $\mathbf{1} = \mathbb{1}$, and inductively $\mathbf{n} + \mathbf{1} = \mathbf{n} + \mathbb{1}$, and $\mathbf{N}(\mathcal{A})$ to be the semiring[†] $\{\mathbf{n} : n \in \mathbb{N}\} \subseteq \mathcal{A}$. When \mathcal{A} is understood we write \mathbf{N} for $\mathbf{N}(\mathcal{A})$.

The most important quasi-zero in a semiring[†] is e , which acts similarly to \emptyset . But e need not absorb in multiplication! Rather:

Remark 1.21. $e^2 = ((\mathbb{1}(-)\mathbb{1})^2 = (\mathbb{1}(-)\mathbb{1}) + (\mathbb{1}(-)\mathbb{1}) = e^\circ = \mathbf{2}e$.

1.4. The main tropical examples.

To prepare for the general algebraic theory, let us review some of the structures that have played a major role so far in tropical algebra.

1.4.1. The max-plus algebra.

The parent structure in tropical algebra is the well-known **max-plus algebra**. We append the subscript \max to indicate the corresponding max-plus algebra, e.g., \mathbf{N}_{\max} or \mathbf{Q}_{\max} . But to emphasize the algebraic structure we still use the usual algebraic notation of \cdot and $+$ throughout, even for the max-plus algebra. The max-plus algebra really concerns ordered groups, such as $(\mathbb{Q}, +)$ or $(\mathbb{R}, +)$, which are viewed at once as max-plus semifields[†], generalizing to the following elegant observation of Green:

Remark 1.22. (i) Any ordered monoid (\mathcal{M}, \cdot) gives rise to a bipotent semiring[†], where we define $a + b$ to be $\max\{a, b\}$. Indeed, associativity is clear, and distributivity follows from the inequalities (1.2).

(ii) Conversely, any semigroup \mathcal{M} has a natural partial pre-order given by $a_1 \geq a_2$ in \mathcal{M} if $a_1 = a_2 + b$ for some $b \in \mathcal{M}$. It is a pre-order when \mathcal{M} is bipotent.

(One could tighten this correspondence by considering lattice-ordered monoids as in [10, 53, 69], but this would take us too far afield here.)

Remark 1.23. The max-plus algebra can be viewed as the triple $(\mathcal{A}, \mathcal{T}, (-))$ of the first kind where $\mathcal{T} = \mathcal{A}$, $(-)$ is the identity, and \preceq is equality, but then $a = (-)a = a^\circ$, which is too crude for us, and we search for alternatives.

1.4.2. Supertropical semirings[†] and supertropical domains[†].

To remedy Remark 1.23, we recall briefly some basics of supertropical algebra.

Definition 1.24. A ν -semiring[†] is a quadruple $R := (R, \mathcal{T}, \mathcal{G}, \nu)$ where R is a semiring[†], \mathcal{T} is a submonoid, and $\mathcal{G} \subset R$ is a semiring[†] ideal, with a multiplicative monoid homomorphism $\nu : R \rightarrow \mathcal{G}$, satisfying $\nu^2 = \nu$ as well as the condition:

$$a + b = \nu(a) \quad \text{whenever} \quad \nu(a) = \nu(b).$$

R is called a **supertropical semiring[†]** when ν is onto, \mathcal{G} is ordered, and

$$a + b = a \quad \text{whenever} \quad \nu(a) > \nu(b).$$

(In this paper we focus on supertropical semirings[†], but the more general definition of ν -semiring[†] enables one to work with polynomials and matrices.)

The elements of \mathcal{G} are called **ghost elements** and $\nu : R \rightarrow \mathcal{G}$ is called the **ghost map**. \mathcal{T} is the monoid of **tangible elements**, and encapsulates the tropical aspect. Here we take $(-)a = a$, a negation map of the first kind.

Definition 1.25. A supertropical semiring[†] R is called a **supertropical domain[†]** when the multiplicative monoid (R, \cdot) is commutative, $\nu|_{\mathcal{T}}$ is 1:1, and R is \mathcal{T} -cancellative.

In this case $\nu|_{\mathcal{T}} : \mathcal{T} \rightarrow \mathcal{G}$ is a monoid isomorphism, and \mathcal{T} inherits the order from \mathcal{G} . In this case, the **standard supertropical semifield[†]** is $\mathcal{T} \cup \mathcal{T}^\nu$ (where customarily $\mathcal{T} = \mathbb{Q}_{\max}$ or \mathbb{R}_{\max}). Addition is now given by

$$a + b = \begin{cases} \nu(a) & \text{whenever } a = b, \\ a & \text{whenever } a > b, \\ b & \text{whenever } a < b. \end{cases}$$

R is called a **supertropical semiring[†]** when ν is onto, \mathcal{G} is ordered, and

$$a + b = a \quad \text{whenever} \quad \nu(a) > \nu(b).$$

The **standard supertropical semifield[†]** is the standard supertropical semifield[†] with 0 adjoined.

We can write R^ν in place of \mathcal{G} , which may be more suggestive. When dealing with supertropical domains we expand both \mathcal{T} and \mathcal{G} to include the same 0 . In other words, $0^\nu = 0$.

We define $e = 1_R + 1_R = 1_R^\nu$. Thus, e is the multiplicative unit of \mathcal{G} , and $R^\nu = eR$.

Conversely, if $e = 1_R + 1_R$ is an additive and multiplicative idempotent of a semiring[†] R , then one can define $\mathcal{G} = Re$ and the projection $\nu : R \rightarrow \mathcal{G}$ given by $r \mapsto re$, thereby recovering the ν -semiring[†] structure.

Remark 1.26. As observed by Knebusch, any module M defined over a supertropical domain[†] itself inherits a map $\nu : M \rightarrow M$ given by $a^\nu = ea$.

Module homomorphisms send ghosts to ghosts, since

$$f(a^\nu) = f(ea) = ef(a) = f(a)^\nu.$$

1.4.3. The standard $(-)$ -supertropical semifield[†].

One can modify Definition 1.25 in the presence of a negation map $(-)$.

Definition 1.27. The **standard $(-)$ -supertropical semifield** is a generalization of the standard supertropical semifield, defined the same way but with addition now given as:

$$a + b = \begin{cases} a & \text{whenever } a > (-)b, \\ a^\circ & \text{for } b = (-)a. \end{cases} \quad (1.6)$$

The supertropical case can be viewed as a special case, when one takes $(-)a = a$ (i.e., $(-)$ of the first kind) and $a^\circ = a + a = a^\nu$. One might expect the standard $(-)$ -supertropical semifield to have the same theory as the standard supertropical semifield, but as we shall see, there are significant differences for negation maps of the second kind.

1.4.4. Layered semirings[†].

“Layered semirings” are described in [40], also cf. [3, Proposition-Definition 2.12]. They are of the form $L \times \mathcal{G}$, where L is the “layering semiring” and (\mathcal{G}, \cdot) is an ordered monoid. The motivating example arises from a valuation $L \rightarrow \mathcal{G}$, so this contains extra information for tropicalization, to be discussed later. We will also consider more generally when \mathcal{G} is a \circ -ordered monoid with a negation map.

Example 1.28. We assume that the “layering semiring” L has a negation map that we designate as $-$. We can define the **layered semiring[†]** as follows:

$\mathcal{A} = L \times \mathcal{G}$. Multiplication is defined componentwise. Addition is given by:

$$(\ell_1, a_1) + (\ell_2, a_2) = \begin{cases} (\ell_1, a_1) & \text{if } a_1 > a_2; \\ (\ell_2, a_2) & \text{if } a_1 < a_2; \\ (\ell_1 + \ell_2, a_1) & \text{if } a_1 = a_2. \end{cases} .$$

Define $e_\ell = (\ell, \mathbb{1})$. Thus $e_1 = \mathbb{1}_{\mathcal{A}} = (1, \mathbb{1}) \in \mathcal{T}$, and by induction, for $\ell \in \mathbb{N}$,

$$e_\ell = e_{\ell-1} + e_1 = \mathbb{1} + \cdots + \mathbb{1},$$

taken ℓ times. Then clearly the e_ℓ generate a sub-semiring with negation map, and $\mathcal{A} = \cup_\ell \mathcal{T}e_\ell$.

Example 1.29. Here are some natural explicit examples of layered semirings:

- (i) $L = \mathbb{N}$ with $\mathcal{T} = \{(\ell, a) \in L \times \mathcal{G} : \ell = 1\}$, and $(-)$ is the identity (thus of the first kind). \mathcal{T}° is the layer 2. (The higher levels, if they exist, are neither tangible nor in \mathcal{T}° when $e' \neq e$. In fact $e' = \mathbb{1} + \mathbb{1} + \mathbb{1}$ has layer 3.)
- (ii) Take $L = \mathbb{N}_0$ in (i), and formally adjoin $\{0\}$ at level 0, to be tangible.
- (iii) $L = \mathbb{Z}$ with the usual negation, $\mathcal{T} = \{(\ell, a) \in L \times \mathcal{G} : \ell = \pm 1\}$, and $(-)(\ell, a) = (-\ell, a)$, of the second kind.
- (iv) L is the residue ring of a valuation, where now $\mathcal{T} = \{(\ell, a) \in L \times \mathcal{G} : \ell \neq 0\}$, and $(-)(\ell, a) = (-\ell, a)$.
- (v) L is a finite field of characteristic 2, where $\mathcal{T} = \{(\ell, a) \in L \times \mathcal{G} : \ell \neq 0\}$, and $(-)$ is the identity (thus of the first kind). This has several interesting theoretical properties, to be specified in Example 6.19, in the context of meta-tangible systems.
- (vi) A somewhat more esoteric example from the tropical standpoint, but quite significant algebraically. Fixing $n > 0$, taking $L = \mathbb{Z}_n$, identify each level modulo n . (This has height n and characteristic n , cf. Definition 5.10.)
- (vii) (Cycling) A weird example, which must be confronted. Fixing $n > 0$, take $L = \{0, \dots, n\}$, but with addition and multiplication given by identifying every number greater than n with n . In other words,

$$k_1 + k_2 = n \text{ in } L \text{ if } k_1 + k_2 \geq n \text{ in } L;$$

$$k_1 k_2 = n \text{ in } L \text{ if } k_1 k_2 \geq n \text{ in } L;$$

There are two candidates for the negation map:

- (First kind) The negation map is the identity.
- (Second kind, for $n > 2$) The negation map sends level ℓ to layer $n - \ell$.

The triple has characteristic 0, since \mathbf{m} does not return to $\mathbb{1}$, but it has height n .

Example 1.30. When \mathcal{G} already has a negation map $(-)$ of second kind we say $a_1 <_{\circ} a_2$ when $a_1^{\circ} < a_2^{\circ}$, and can define addition more generally by:

$$\begin{aligned} (\ell_1, a_1) + (\ell_2, a_2) &= \begin{cases} (\ell_1, a_1) & \text{if } a_1 >_{\circ} a_2; \\ (\ell_2, a_2) & \text{if } a_1 <_{\circ} a_2; \\ (\ell_1 + \ell_2, a_1) & \text{if } a_1 = a_2; \\ (\ell_1 + \ell_2, a_1^{\circ}) & \text{if } a_1 = (-)a_2; \end{cases} \\ (\ell_1, a_1^{\circ}) + (\ell_2, a_2) &= \begin{cases} (\ell_1, a_1^{\circ}) & \text{if } a_1 >_{\circ} a_2; \\ (\ell_2, a_2) & \text{if } a_1 <_{\circ} a_2; \\ (\ell_1 + \ell_2, a_1 e') & \text{if } a_1 = (\pm)a_2. \end{cases} \\ (\ell_1, a_1^{\circ}) + (\ell_2, a_2^{\circ}) &= \begin{cases} (\ell_1, a_1^{\circ}) & \text{if } a_1 >_{\circ} a_2; \\ (\ell_2, a_2^{\circ}) & \text{if } a_1 <_{\circ} a_2; \\ (\ell_1 + \ell_2, a_1^{\circ} + a_2^{\circ}) & \text{if } a_1 = (\pm)a_2. \end{cases} \end{aligned}$$

(This reduces to the previous case when the negation map on \mathcal{G} is the identity.)

1.4.5. Symmetrized semirings[†].

(The next construction also works for \mathcal{T} -semirings, cf. Definition 2.13 below), to handle the hyperring case.)

Definition 1.31. Given any semiring[†] R , define $\hat{R} = R \times R$ with componentwise addition and “twisted” multiplication

$$(r_0, r_1)(r'_0, r'_1) = (r_0 r'_0 + r_1 r'_1, r_0 r'_1 + r_1 r'_0).$$

We call \hat{R} the **symmetrized semiring[†]** of R .

Lemma 1.32. If R is a semiring, then \hat{R} is a semiring under the twist action.

Proof. \hat{R} is clearly a module, and one checks associativity of multiplication and distributivity. \square

This is the structure given at the beginning of [27, §3.8] in the case that $R = \mathbb{R}_{\max}$, and is the venue for [2], [6, §3.4], [3, Example 2.21], and [8, 52]), rather than what is called the “symmetrized algebra” in [27]. But we prefer the terminology “symmetrized” for this version, which is appropriate to the general structure theory.

1.4.6. The symmetrized version according to [3, Proposition-Definition 2.12].

The following construction was introduced in [2, Proposition 5.1] and explored further under the name of “symmetrized max-plus semiring” in [3, Proposition-Definition 2.12], as an alternate way of viewing some tropical constructions.

Example 1.33. One starts with an ordered semigroup \mathcal{G} , putting $\mathcal{G}_0 = \mathcal{G} \cup \{0\}$, and the layered semiring $\mathcal{A} = L \times \mathcal{G}_0$, where $L = \mathcal{G}_0$, and we define

$$\mathcal{G}_{\text{sym}} := (\mathcal{G} \times \{0\}) \cup (\{0\} \times \mathcal{G}) \cup \{(a, a) : a \in \mathcal{G}_0\} \subseteq \mathcal{A}.$$

Thus, viewing \mathcal{G}_0 as a semiring, addition on $\mathcal{G} \times \{0\}$, $\{0\} \times \mathcal{G}$, and $\{(a, a) : a \in \mathcal{G}_0\}$ is according to components, whereas “mixed” addition satisfies:

$$\begin{aligned} (a_0, 0) + (0, a_1) &= \begin{cases} (a_0, 0) & \text{if } a_0 > a_1; \\ (0, a_1) & \text{if } a_0 < a_1; \\ (a_1, a_1) & \text{if } a_0 = a_1; \end{cases} \\ (a_0, 0) + (a_1, a_1) &= \begin{cases} (a_0, 0) & \text{if } a_0 > a_1; \\ (a_1, a_1) & \text{if } a_0 \leq a_1; \end{cases} \\ (0, a_0) + (a_1, a_1) &= \begin{cases} (0, a_0) & \text{if } a_0 > a_1; \\ (a_1, a_1) & \text{if } a_0 \leq a_1. \end{cases} \end{aligned}$$

This particular semiring also has the negation map given by the “switch” $(-)(a_0, a_1) = (a_1, a_0)$ and has very nice properties, that is best explained in terms of its ensuing system.

1.4.7. Quasi-classical algebras.

Here is an intriguing combination of these max-plus related algebras with classical algebra, which will take a prominent role in §6.9.

Example 1.34. *We start with a classical algebra R , such as \mathbb{Z} , which acts on another semiring \mathcal{M} in the sense that \mathcal{M} is an R -module (for example a max-plus algebra), which we view as “infinitesimals,” in the sense that $r + a = r$ for all $r \in R \setminus \{0\}$ and $a \in \mathcal{M}$. Thus addition projects onto the classical part, which implies that we have associativity and distributivity (since any term from R enables us to ignore the part from \mathcal{M} , and if there is no term from R we use associativity and distributivity from \mathcal{M}).*

1.5. Optional: Hypergroups.

Recent interest has arisen in the study of hypergroups and hyperrings, in particular hyperfields, cf. [74, 51, 7]. It turns out that the hypergroups of [7, 74] can be injected naturally into their power sets, which have a negation map, whereby the hyperfield is identified with the subset of singletons. We have labeled the hypergroup material as optional because it involves extra complications (such as \mathcal{A} not necessarily being distributive), but it is recommended since hypergroup theory has inspired much of the material in systems. We basically follow the treatments of Baker [7] and Jun[51].

The idea is to formulate all of our extra structure in terms of addition (and possibly other operations) on $\mathcal{P}(\mathcal{T})$, the set of subsets of \mathcal{T} , viewed as an additive semigroup or even a module over \mathcal{T} when $(\mathcal{T}, \cdot, 1)$ is a monoid. But this is not so easy since $\mathcal{T}_0 \cup \{0\}$ itself need not be closed under addition.

The “intuitive” definition: A hyper-semigroup should be a structure $(\mathcal{T}, \boxplus, 0)$ where $\boxplus : \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{P}(\mathcal{T})$, for which the analog of associativity holds:

$$(a_1 \boxplus a_2) \boxplus a_3 = a_1 \boxplus (a_2 \boxplus a_3), \quad \forall a \in \mathcal{T}.$$

There is a fundamental difficulty in this definition: $a_1 \boxplus a_2$ need not be a singleton, so technically $(a_1 \boxplus a_2) \boxplus a_3$ is not defined. This difficulty is exacerbated when considering generalized associativity; for example, what does $(a_1 \boxplus a_2) \boxplus (a_3 \boxplus a_4)$ mean in general? We rectify this by passing to the power set $\mathcal{P}(\mathcal{T})$.

Definition 1.35. A *hyper-semigroup* is $(\mathcal{T}, \boxplus, 0)$, where

- (i) \boxplus is a commutative binary operation $\mathcal{T} \times \mathcal{T} \rightarrow \mathcal{P}(\mathcal{T})$, which also is associative in the sense that if we define

$$a \boxplus S = \cup_{s \in S} a \boxplus s,$$

then $(a_1 \boxplus a_2) \boxplus a_3 = a_1 \boxplus (a_2 \boxplus a_3)$ for all a_i in \mathcal{T} .

- (ii) 0 is the neutral element.

We write $\hat{\mathcal{T}}$ for $\{\boxplus a_i : a_i \in \mathcal{T}\}$. Note that $\{a\} = a \boxplus 0 \in \hat{\mathcal{T}}$. Thus there is a natural embedding $\mathcal{T} \hookrightarrow \hat{\mathcal{T}}$ given by $a \mapsto \{a\}$, and we can transfer the addition by defining

$$\{a_1\} \boxplus \{a_2\} = a_1 \boxplus a_2.$$

We always think of \boxplus in terms of addition. Note that repeated addition in the hyper-semigroup is not defined until one passes to the power set, which makes it difficult to check basic universal relations such as associativity. Associativity could hold at the level of elements but fail at the level of sets.

Definition 1.36. A *hypernegative* of an element a in a hyper-semigroup $(\mathcal{T}, \boxplus, 0)$ is an element $-a$ for which $0 \in a \boxplus (-a)$.

(Following [51, Definition 2.1]) A *hypergroup* is a hyper-semigroup $(\mathcal{T}, \boxplus, 0)$ for which every element a has a unique hypernegative, designated $-a$. A *canonical hypergroup* is a hypergroup satisfying the extra property:

- (reversibility) $a \in b \boxplus c$ iff $c \in a \boxplus (-b)$.

Viro [74, Definition 3.1] calls this a **multigroup**.

Remark 1.37. Henry [35, §2] shows that unique hypernegatives together with associativity as defined above implies the reversibility condition. Thus, the term “canonical” is superfluous.

Lemma 1.38. *If $(\mathcal{T}, \cdot, \mathbb{1})$ is a semigroup and $(\mathcal{T}, \boxplus, 0)$ is a hyper-semigroup, then \mathcal{T} acts on $\widehat{\mathcal{T}}$ via the action*

$$aS = \{as : s \in S\}. \quad (1.7)$$

Proof. $(a_1 a_2)S = \{(a_1 a_2)s : s \in S\} = \{a_1(a_2 s) : s \in S\} = a_1(a_2 S)$. □

A \mathcal{T} -**hyperzero** of a hyper-semigroup $(\mathcal{T}, \boxplus, 0)$ is a set of the form $a \boxplus (-a) \in \mathcal{P}(\mathcal{T})$. (This is not the usual definition, which is any subset of \mathcal{T} containing 0, but serves just as well since, by definition, if $0 \in a \boxplus b$ for $a, b \in \mathcal{T}$ then $b = -a$, implying $a \boxplus b$ is a hyperzero in our sense.)

Definition 1.39. A **hypermodule** over a monoid $(\mathcal{T}, \cdot, \mathbb{1})$ is a hyper-semigroup $(\mathcal{S}, \boxplus, 0)$ together with an action of \mathcal{T} on \mathcal{S} such that distributivity holds for $\widehat{\mathcal{S}}$ over \mathcal{T} .

$(\mathcal{T}, \boxplus, \cdot, \mathbb{1})$ is a **hyperring** if $(\mathcal{T}, \boxplus, 0)$ also is a hypermodule over $(\mathcal{T}, \cdot, \mathbb{1})$.

A hyperring $(\mathcal{T}, \boxplus, \cdot, \mathbb{1})$ is a **hyperfield** if $(\mathcal{T}, \cdot, \mathbb{1})$ is a group.

In other words, a hyperfield is $(\mathcal{T}, \boxplus, \cdot, 0, \mathbb{1})$ where $(\mathcal{T}, \cdot, \mathbb{1})$ is a group and

- (i) \boxplus is an associative binary operation $\mathcal{T} \times \mathcal{T} \rightarrow \mathcal{P}(\mathcal{T})$;
- (ii) 0 is the neutral element, in the sense that $g \boxplus 0 = 0 \boxplus g = \{g\}$, $\forall g \in \mathcal{T}$;
- (iii) For any $a \in \mathcal{T}$ there is a unique element $-a \in \mathcal{T}$ such that $0 \in a \boxplus (-a)$.

Jun [51, Definition 2.3] defines a **hypergroup morphism** to be a map $f : \mathcal{T}_1 \rightarrow \mathcal{T}_2$ of hypergroups, satisfying $f(a \boxplus b) \subseteq f(a) \boxplus f(b)$.

1.5.1. Negation maps on hypergroups.

Lemma 1.40. *The hyper-negation on a hypergroup (resp. hyperring) \mathcal{T} is a negation map, and induces a negation map on $\mathcal{P}(\mathcal{T})$, via $(-)S = \{-s : s \in S\}$.*

Proof. To see that $-(a_1 + a_2) = (-a_1) \boxplus (-a_2)$, note that $0 \in a_i \boxplus (-a_i)$ for $i = 1, 2$, so

$$0 \in a_1 \boxplus (-a_1) \boxplus a_2 \boxplus (-a_2) = (a_1 \boxplus a_2) \boxplus (-a_1) \boxplus (-a_2).$$

Likewise, $0 \in a_1 \boxplus (-a_1)$ implies $0 \in a(a_1 \boxplus (-a_1)) = aa_1 \boxplus (-aa_1)S$.

In case \mathcal{T} is a hyperring we note from the previous paragraph that $-(a_1 a_2) = (-a_1)a_2 = a_1(-a_2)$, and thus $(-a_1)(-a_2) = -(-a_1)a_2 = a_1 a_2$. □

1.6. Introducing surpassing relations.

The next observation is quite relevant since tropical algebra starts with the max-plus algebra, which then is refined to permit the use of more sophisticated algebraic techniques.

Example 1.41. *We have three main partial pre-orders in a semigroup M , according to which environment we find ourselves:*

- (i) *(For any semigroup \mathcal{A}) Green's relation \geq of Remark 1.22.*
- (ii) *The **ghost-surpassing relation** $a_1 \models_g a_2$ in \mathcal{A} if $a_1 = a_2 + b$ for some $b \in \mathbf{2}\mathcal{A}$. (This pertains in particular to the supertropical theory, cf. Definition 1.24ff., since $\mathbb{1}_R + \mathbb{1}_R = e$ in this case.)*
- (iii) *(When \mathcal{A} has a negation map) The **o-surpassing relation** $a_1 \succeq_o a_2$ in \mathcal{A} if $a_1 = a_2 + b$ for some $b \in \mathcal{A}^\circ$.*

Any such relation becomes trivial when \mathcal{A} contains $-\mathbb{1}$, the negative of $\mathbb{1}$, since then $a_2 = a_1 + ((-\mathbb{1})b)$.

Here is an enlightening example of how (iii) generalizes classical algebra. In any semiring with negation map, we write $[a, b]$ for the **Lie commutator** $ab(-)ba$.

Lemma 1.42 (Leibniz \preceq -identities). $[a, b]c + b[a, c] = [a, bc] + (bac)^\circ$. In particular,

$$[a, b]c + b[a, c] \succeq_o [a, bc]; \quad a[b, c] + [a, c]b \succeq_o [ab, c].$$

Proof. $[a, b]c + b[a, c] = (ab(-)ba)c + b(ac(-)ca) = (abc(-)bca) + (bac(-)bac) = [a, bc] + (bac)^\circ$. The second assertion is analogous. □

Example 1.41 is tied in with the following property:

Definition 1.43. A semigroup $S \subseteq \mathcal{A}$ is **ub** (for **upper bound**) if $a + b + c = a$ always implies $a + b = a$.
 \mathcal{A} is **\mathcal{T} -ub** if $a + b + c = a$ for $b, c \in \mathcal{T}$ always implies $a + b = a$.
 \mathcal{A} is **\circ -ub** if $a + b + c = a$ for $b, c \in \mathcal{A}^\circ$ always implies $a + b = a$.

This criterion abounds in tropical algebra, as noted in [42]. (On the other hand, it fails miserably in classical algebra.) For example, the max-plus algebra is a ub semifield[†]. The point of ub is seen in Remark 1.44 (also see [42, Proposition 3.10]).

Any polynomial semiring[†] or matrix semiring[†] over a ub (resp. \mathcal{T} -ub, \circ -ub) semiring[†] is ub (resp. \mathcal{T} -ub, \circ -ub).

Remark 1.44. By [44, Proposition 0.5], the partial pre-order (i) of Example 1.41 is a PO iff the semigroup \mathcal{A} is ub.

Partial pre-order (ii) lies at the heart of much of [39, 40, 41, 42, 43, 44, 45, 46, 47, 49], where again it is a PO. For R a supertropical semiring[†] (Definition 1.24ff.), the partial pre-order (ii) can be viewed as a special case of (iii), where one defines $(-)a = a$; however, the two can be distinct within the same model. (One can start with any ν -semiring[†] and then symmetrize.)

Partial preorder (iii) is the focus of this paper, and as Blacher [12] points out, is a partial order iff \mathcal{A} is \circ -ub. Indeed, if $a \succeq b$ and $b \succeq a$, then $a = b + c^\circ$ and $b = a + d^\circ$, implying $b = b + c^\circ + d^\circ$, and thus $b = b + c^\circ = a$.

The ensuing theory involves some subtleties which cannot be explained properly until we formally incorporate a “surpassing relation” into the structure.

Definition 1.45. A **surpassing relation**, denoted \preceq , is a partial pre-order satisfying the following, for elements of \mathcal{A} :

- (i) $a \preceq b$ whenever $a + c^\circ = b$ for some $c \in \mathcal{A}^\circ$.
- (ii) If $a \preceq b$ then $(-)a \preceq (-)b$.
- (iii) If $a_i \preceq b_i$ for $i = 1, 2$ then $a_1 + a_2 \preceq b_1 + b_2$.
- (iv) If $a \preceq b$ for $a, b \in \mathcal{T}$, then $a = b$.
- (v) If $a^\circ \preceq b$ for $b \in \mathcal{T}$, then $b = \mathbb{0}$.
- (vi) \preceq restricts to a PO on \mathcal{A}° .

Lemma 1.46. If $a \preceq b + b'$ and $b \preceq c + c'$, then $a \preceq c + (b' + c')$.

Proof. $a \preceq b + b' \preceq c + c' + b' = c + (b' + c')$. □

Let us pause to see why the conditions of Definition 1.45 are needed for \preceq to parallel equality.

- (i) shows that \preceq refines \preceq_\circ , defined presently, and shows how the quasi-zeros behave like $\mathbb{0}$ under \preceq .
- (ii), (iii) are needed for considerations in universal algebra.
- (iv) enables us to view \preceq as equality for tangible elements.
- (v) gives an important relation among the quasi-zeros.

Definition 1.47. The \circ - **relation** \preceq_\circ is the \circ -surpassing relation of Example 1.41(iii), namely $a \preceq_\circ b$ iff $b = a + c$ for some $b \in \mathcal{A}^\circ$.

Some easy general observations:

Lemma 1.48.

- (i) If $a \preceq_\circ b$ then $(-)a \preceq_\circ (-)b$.
- (ii) $(-)a \preceq_\circ c$ implies $a + c \in \mathcal{A}^\circ$.

Proof. (i) $a + c^\circ = b$ implies $(-)a + c^\circ = (-)(a + c^\circ) = (-)b$.

(ii) Write $c = b^\circ(-)a$. Then $a + c = a^\circ + b^\circ \in \mathcal{A}^\circ$. □

One can check that \preceq_\circ is indeed a surpassing relation in the examples we just gave (Definition 1.25) and Examples 1.27, 1.34, and 1.28. More generally, for any meta-tangible triple, we shall see in Theorem 6.29 that the relation \preceq_\circ satisfies the conditions of Definition 1.45.

One other property that one would like is that $a \preceq_\circ a^\circ$, which holds in all of these tropical examples except the layered (when $e' \neq e$), and fails miserably in the classical case.

The other motivating example of a surpassing relation is given in our discussion of hypergroups in §5.7.

Recall the **quasi-zero** $a^\circ := a(-)a$. In classical algebra, the only quasi-zero is 0 itself. In the supertropical theory, the quasi-zeros are the “ghost” elements. In [2] the quasi-zeros have the form (a, a) . At any rate, the set of quasi-zeros replaces 0 in our theory. For example, a **quasi-negative** of $a \in \mathcal{T}$ is an element $b \in \mathcal{T}$ such that $a + b$ is a quasi-zero.

By definition, $(-)a$ is a quasi-negative of a .

Definition 1.49. \mathcal{T} has **unique quasi-negatives** if $a_1 + a_2 \in \mathcal{A}^\circ$ for $a_i \in \mathcal{T}$ implies $a_2 = (-)a_1$.

1.7. Introducing systems.

We begin to put everything together. We are interested in the **system**[†] $(\mathcal{A}, \mathcal{T}, (-), \preceq)$, given precisely in Definition 5.1, where \mathcal{A} is defined via some signature in universal algebra that includes addition, \mathcal{T} (occasionally denoted $\mathcal{T}(\mathcal{A})$ when \mathcal{A} is ambiguous) is a distinguished subset of \mathcal{A} , called the “tangible elements,” that generates $(\mathcal{A}, +)$, $(-)$ is a negation map satisfying $(-)\mathcal{T} = \mathcal{T}$, and \preceq is an appropriate surpassing relation. Often \mathcal{A} is a semiring[†] (resp. semifield[†]), and \mathcal{T} is taken to be a multiplicative submonoid (resp. subgroup) of \mathcal{A} .

A **system** is a system[†] perhaps with 0 adjoined. The system is of **k -th kind** (for $k = 1, 2$) when $(-)$ is of k -th kind.

Almost always, our principal interest is in \mathcal{T} , the focus of the applications, although \mathcal{T} is best understood in terms of its relation to \mathcal{A} . Versions of systems come up in varied situations, such as in tropical algebra and hypergroups, which “explains” similarities in their theories. The following property is pervasive in tropical algebra, and comes to the fore in Theorem 6.18. We start with the triple $(\mathcal{A}, \mathcal{T}, (-))$ and then bring in the surpassing relation \preceq .

Definition 1.50. The triple $(\mathcal{A}, \mathcal{T}, (-))$ is **uniquely negated** if \mathcal{T} has unique quasi-negatives.

Uniquely negated triples already satisfy some nice properties, such as $\mathcal{T} \cap \mathcal{A}^\circ = \{0\}$ (Corollary 5.3).

Any (uniquely negated) triple containing 1 has the (uniquely negated) sub-triple $\langle 1 \rangle$ generated by 1 , which plays a role parallel to \mathbf{F}_1 -geometry. $\langle 1 \rangle$ contains \mathbf{N} , e , and e' , and plays an important role in many proofs, since $a + a = 2a$, $a(-)a = ae$ and $a(-)a + a = ae'$. In particular, if $2 = 1$, then $a + a = 2a = a$, so \mathcal{A} is idempotent. (See Remark 6.2 below for a more thorough explanation.) We are interested in particular in whether or not $\langle 1 \rangle$ has height 2, which largely depends on the value of e' .

We can hone the theory further to the following types of triple:

Definition 1.51. A triple $(\mathcal{A}, \mathcal{T}, (-))$ is **meta-tangible** if $\mathcal{T}_0 + \mathcal{T}_0 \subseteq \mathcal{T} \cup \mathcal{A}^\circ$. A **meta-tangible system** is a uniquely negated system $(\mathcal{A}, \mathcal{T}, (-), \preceq)$, for which the triple $(\mathcal{A}, \mathcal{T}, (-))$ is meta-tangible.

A special case: $(\mathcal{A}, \mathcal{T}, (-))$ is **$(-)$ -bipotent** if $a + b \in \{a, b\}$ whenever $a, b \in \mathcal{T}$ with $b \neq (-)a$. In other words, $a + b \in \{a, b, a^\circ\}$ for all $a, b \in \mathcal{T}$. (We also say that \mathcal{T} is **$(-)$ -bipotent**.)

In Theorem 6.18 we shall see that meta-tangible triples are “almost” bipotent. The classification of meta-tangible systems in Theorem 6.57, shows how the major tropical examples appear in terms of the axiomatic theory.

Although $(-)$ -bipotent triples are the most significant from the tropical standpoint, they do not apply to some of the more important hyperfields such as the phase hyperfield and the triangle hyperfield.

1.7.1. Important examples of systems.

Our main goal is to lift the classical theory to other systems (often meta-tangible) with a more tropical flavor. These systems are surprisingly ubiquitous, and enable us to classify many related notions. Often the role of $(-)1$ is critical. Their thrust is summarized in the following collection of examples.

Before delving into the theory, we consider some of the main examples, even though the details are only given in what follows. Systems[†] (or systems, when containing 0) $(\mathcal{A}, \mathcal{T}, (-), \preceq)$, can be described according to various properties, with some corresponding examples:

Example 1.52.

- (i) *Systems arising from semirings.*
 - (a) *Height 1. This makes $\mathcal{T}_0 = \mathcal{A}$.*

- *Classical algebra*, by which we mean that $(\mathcal{A}, +)$ is a group. Here $\mathcal{A}^\circ = \{0\}$, so the quasi-negative is the usual negative, which is unique. $a \preceq_\circ b$ iff $b = a + 0 = a$, so we have the system $(\mathcal{A}, \mathcal{T}, -, =)$, which is meta-tangible.
In some ways we want the general theory of meta-tangible systems to mimic classical algebra. But one big difference is that in classical algebra $a^\circ = 0 = b^\circ$ for all a, b ; see Definition 6.39. Also, negation is of second kind unless A has characteristic 2, in which case $(-)$ is of the first kind. This helps to “explain” why the theory of meta-tangible systems of first kind often has the flavor of characteristic 2.
 - Taking $\mathcal{T} = \mathcal{A} = \mathcal{A}^\circ$ yields the max-plus algebra, but the quasi-negatives are far from unique, since whenever $b < a$ we have $a + b = a = a^\circ$. (This is one reason why we shy away from the max-plus algebra in our algebraic theory.) Here $a^\circ = b^\circ$ implies $a = b$, cf. Definition 6.39.
- (b) *Height 2*. These provide more refined tropical structures, designed to improve the max-plus algebra. All of them are $(-)$ -bipotent systems, which are studied in depth. The familiar examples have characteristic 0 (Definition 5.10), although the constructions can also be replicated in positive characteristic.
- Supertropical semirings[†], cf. Definition 1.24, can be described as the $(-)$ -bipotent systems $(\mathcal{A}, \mathcal{T}, (-), \preceq)$ of the first kind and height 2, where \mathcal{T} is the set of tangible elements, and $a^\circ = a + a = a^\nu$.
 - The “symmetrized” system of Example 1.33, is $(-)$ -bipotent of the second kind, since $(-)(a, 0) = (0, a)$, and their sum is (a, a) (but all other sums of elements of \mathcal{T} are taken from the maximum).
(Incidentally, the system of Definition 1.31, where $\mathcal{T}(\hat{\mathcal{A}}) = \mathcal{T} \times \{0\} \cup \{0\} \times \mathcal{T}$, and $(-)$ is the “twist action” $(-)(a_1, a_2) = (a_2, a_1)$, with \preceq as in Example 1.41(iii), may be more intuitive at first glance, but it is not uniquely negated, which might have been what led Gaubert to Example 1.33.)
 - The “exploded” system of Example 1.29(iii), where $\mathcal{A} = L \times \mathcal{G}$ with L the set of lowest coefficients of Puiseux series, $\mathcal{T} = (L \setminus 0) \times \mathcal{G}$ and $(-)(\ell, a) = (-\ell, a)$, is $(-)$ -bipotent of the second kind, provided L is not of characteristic 2.
- (c) *Height ≥ 3* . The “layered” system of Example 1.29(i), which was designed to handle derivatives, is $(-)$ -bipotent of the first kind. It has height equal to the cardinality of the submonoid of L generated by 1, which often has height 3 and even could be of infinite height. It often provides counterexamples to assertions which hold in height 2.
- (ii) *Systems arising from a hypergroup \mathcal{T}* . These all arise from $(\mathcal{P}(\mathcal{T}), \mathcal{T}, -, \preceq)$, where $\mathcal{P}(\mathcal{T})$ is the power set, $-$ is the hypernegative, and $S_1 \preceq S_2$ iff $S_1 \subseteq S_2$. There are some subtleties treated in §5.7. See Example 12.8 below for some of the main examples. The “canonical” hypergroups yield uniquely negated systems, by definition, but other properties may differ:
- (a) $(-)$ -Bipotent hypergroups include Viro’s “tropical hyperfield,” which is isomorphic to the tangible part of the supertropical algebra, the Krasner hypergroup (of the first kind), and the sign hypergroup (of the second kind), all of height 2.
 - (b) The phase hypergroup, of the second kind, is non- $(-)$ -bipotent but idempotent of height 3. Here $\mathbb{1}$ is the point that is one radian along the circle, and $e' = e = \mathbb{1}(-)\mathbb{1}$ is the arc from $(-)\mathbb{1}$ to $\mathbb{1}$.
 - (c) Viro’s “triangle” hyperfield of Example 12.8, is of the first kind and not idempotent. Here distributivity holds only with respect to elements of \mathcal{T} .
 - (d) Lopez’ non-canonical hypergroup gives rise to a non-uniquely negated triple of the first kind, of height 2, where the tangible elements are the points.
- (iii) Suppose $M = \mathcal{A}^{(I)}$. Then $(-)$ is given by $(-)(a_i) = ((-)a_i)$. One gets a triple, $(\mathcal{A}^{(I)}, \mathcal{T}^{(I)}, (-))$, where $\mathcal{T}^{(I)} := (\mathcal{T}_0)^I \setminus \{(0)\}$, and $(-)$ is taken componentwise.
This triple is not $(-)$ -bipotent for $|I| \geq 2$, since when taking sums, one component could be the sum of quasi-negatives (and thus leave $\mathcal{T}(M)$) whereas the other is not. Nevertheless, they provide the means of studying polynomials.

Another choice, more relevant to this paper: Define \mathcal{T}_i as the canonical injection of \mathcal{T} into the i -th component, and $\mathcal{T}(M) = \cup_{i \in I} \mathcal{T}_i$. The surpassing relation \preceq now can be extended componentwise to $\mathcal{T}(M)$ and is surpassing, seen by checking components. $(\mathcal{A}^{(I)}, \mathcal{T}(M), (-), \preceq)$ is a uniquely negated system.

Matrices over systems are treated in §9.1.1.

- (iv) Tensor products of triples are triples, as described in §8.5. Although they lose $(-)$ -bipotence, they provide a powerful tool in the theory, especially in tropicalization and the ensuing definition (and motivation) of varied tropical structures such as Grassmann semialgebras, super-semialgebras, Lie semialgebras, and Poisson semialgebras, to be discussed in §11.

Remark 1.53. Triples of the first kind behave quite differently from those of the second kind. (This is especially evidenced in linear algebra.)

Triples of the first kind that contain $\mathbb{1}$ satisfy $e' = \mathbb{1} + \mathbb{1} + \mathbb{1}$.

- If $e' = \mathbb{1}$, then we are in characteristic 2.
- If $\mathbb{1} + \mathbb{1} + \mathbb{1} = e = \mathbb{1} + \mathbb{1}$, the system often has height 2, such as in the first two examples of Examples 1.52(i(b)), and when $(-)$ -bipotent is the supertropical domain: \mathcal{T} is the set of tangible elements.
- If $\mathbb{1} + \mathbb{1} + \mathbb{1} \notin \mathcal{T}^+$, then we are in the more esoteric region of height ≥ 3 and layered algebras, cf. Examples 1.52(i(c)) and (ii(b,c)).

Triples of the second kind often have either the flavor of classical algebra or of the symmetrized algebra of [3]. $(-)$ -Bipotent triples of the second kind all are idempotent since $a + a \in \max\{a, a\} = a$. (The converse also holds, as to be seen in Corollary 6.20.) But since $(-)$ -bipotence does not apply to some of the more important hyperfields such as the phase hyperfield and the triangle hyperfield, one is led to weaken this at times to idempotence.

Every $(-)$ -bipotent triple clearly is meta-tangible. Perhaps as a surprise, conversely, by Theorem 6.18, a meta-tangible triple either is $(-)$ -bipotent (with the ensuing tropical flavor) or satisfies $e' = \mathbb{1}$ (in which case $e^\circ = e$), which is the case in classical algebra.

The elements of meta-tangible triples have a surprisingly nice form given in Theorem 6.25, which enables us to prove, with one class of exceptions, that meta-tangible triples are \mathcal{T} -reversible (Theorem 6.38) and often are matroidal (Lemma 6.32, Proposition 6.35), although there are annoying counterexamples (Examples 6.34, 6.37).

The heights of elements lead to considerations of the **characteristic** (Definition 5.10) of a triple, in Theorem 6.26. Theorem 6.50 enables us to characterize the symmetrized algebra in terms of classical considerations about sums of squares. This pertains to “real” groups of tangible elements, in Proposition 6.56.

1.8. Introducing symmetrization.

The next step is to provide a construction with a built-in negation map of the second kind, in order to enhance the tools from classical algebra. Although the max-plus algebra and its modules initially lack negation, one obtains negation maps for them through the next main idea, the symmetrization process of §7.4, again extracted from [2, 26, 32], where an algebraic structure is embedded into a doubled structure with a natural negation map. (As explained in §7.1, this is a special case of super-spaces, i.e., 2-graded structures popularized by the physicists.)

Given a semigroup $(M, +)$, we pass to $\widehat{M} = M \times M$ with componentwise addition. This process is reminiscent of the familiar construction of \mathbb{Z} from \mathbb{N} by taking ordered pairs (a_0, a_1) (intuitively identified with $a_0 - a_1$). Golan [30, Chapter 16] deals with the equivalence identifying (a_0, a_1) with (a'_0, a'_1) when $a_0 + a'_1 = a_1 + a'_0$, and in particular identifying (a, a) with $(0, 0)$. But we must refrain from applying this equivalence, since it degenerates in tropical algebra. Nevertheless, we do have the negation map given by $(a_0, a_1) \mapsto (a_1, a_0)$, which we call the **switch map**. In other words, we treat (a_0, a_1) as $a_0 - a_1$ although we do not make the identification.

Having stopped short of passing to equivalence classes, we need to replace $\{0_M\}$ by the set of quasi-zeros, which is why we utilize surpassing relations.

Lemma 1.54. *If $\mathcal{O}_M \in M$, then $\widehat{M}^\circ = \{(a, a) : a \in M\}$.*

Proof. (\subseteq) $(a_0, a_1)(-)(a_0, a_1) = (a_0 + a_1, a_0 + a_1)$.

(\supseteq) $(a, a) = (a, \mathcal{O}_M)^\circ \in M^\circ$. □

Since $(r_0 - r_1)(a_0 - a_1) = r_0 a_0 + r_1 a_1 - (r_0 a_1 + r_1 a_0)$ in classical algebra, we formally define the new module action $\widehat{R} \times \widehat{M} \rightarrow \widehat{M}$ as in [2] and [8]:

$$(r_0, r_1)(a_0, a_1) = (r_0 a_0 + r_1 a_1, r_0 a_1 + r_1 a_0). \quad (1.8)$$

The following trivial observation becomes a pillar of the theory:

Remark 1.55. *Suppose M is a module over a semiring[†] \mathcal{A} . One can recover M from \widehat{M} by projecting onto the first component. Thus, the correspondence $N \mapsto \widehat{N}$ embeds the lattice of R -submodules of M into the lattice of $\widehat{\mathcal{A}}$ -submodules of $M \times M$ (and these are the congruences on M).*

The \circ -surpassing relation leads to a major application, the **transfer principle**, originating in [65, p. 352, end of proof of (a)], in [26], and obtained for matrices in [2, Theorem 3.4].

A delicate issue here is the natural map from \mathbb{N} to an arbitrary semiring[†] R (or from \mathbb{N}_{+0} to a semiring), which parallels the natural map from \mathbb{Z} to an arbitrary ring.

Lemma 1.56. *There is a semiring[†] homomorphism $\mathbb{N} \rightarrow R$ given by $n \mapsto \mathbf{n}$ (of Definition 1.20).*

Proof. A standard and easy induction. □

Remark 1.57. *The tricky point is that we may not be able to identify n with \mathbf{n} . For example, in the max-plus algebra, $\mathbf{n} = 1$, not n . This leads to the notion of “characteristic 1” and “ \mathbf{F}_1 geometry” and plays an implicit role in our discussion below of transfer. We cope with this issue by defining the semiring \mathbb{N}_n of natural numbers **truncated at n** to be the homomorphic image of \mathbb{N} in which $\mathbf{n} = \mathbf{n} + 1$.*

Here is the transfer principle, which we formulate somewhat more generally in universal algebra, as described in §2.

Theorem 7.19 [Transfer principle, strong form]. We work in an additive signature of universal algebra in which all universal relations generated by \mathcal{I} are \mathbb{N} torsion free, i.e., the natural map $\mathbb{N}\{x; \Omega, \mathcal{I}\} \rightarrow \mathbb{Z}\{x; \Omega, \mathcal{I}\}$ is injective. Suppose that the semiring[†] $\widehat{\mathbb{N}}\{x; \Omega, \mathcal{I}\}$ (under the usual operations of \mathbb{N}) satisfies an identity

$$(P_1, Q_1) \equiv (P_2, Q_2)$$

for $P_i, Q_i \in \mathbb{N}\{x; \Omega, \mathcal{I}\}$ where also $(P_1, Q_1) \geq (P_2, Q_2)$. Then $(P_1, Q_1) \succeq_\circ (P_2, Q_2)$ in $\widehat{\mathbb{N}}_n\{x; \Omega, \mathcal{I}\}$.

Relevant concepts from general algebraic theory are covered in §8, including tensor products (§8.5) and involutions (§5.6)).

In §8 we turn to the category of systems. Some ideas can be formulated best in terms of \preceq -morphisms, cf. §8, in particular Definition 8.4, which by definition respect the surpassing relation \preceq . This fits in well with recent research on hypergroups, [51], and enables one to embed the category of hypergroups into the category of systems (Theorem 8.9).

Here is a sample illustration of how tropical theory can be given a classical flavor. Given a module M over a semiring[†] R , we define $\text{End}_R M$ to be $\text{Hom}(M, M)$, the set of module homomorphisms from M to itself, a semiring under composition of maps.

Proposition 11.17. *If L is a Lie semialgebra (over R) with a negation map, then $\text{ad } L$ is a Lie sub-semialgebra of $\text{End}_R L$, and there is a Lie \preceq -morphism $L \rightarrow \text{ad } L$, given by $a \mapsto \text{ad}_a$.*

The same approach leads us to a variant of identical relation from universal algebra.

Definition 1.58. *A **surpassing identical relation** of an $(\Omega; \mathcal{I})$ -algebra \mathcal{A} is a pair*

$$(f(x_1, \dots, x_n), g(x_1, \dots, x_n))$$

of Ω -formulas such that $f(x_1, \dots, x_n) \geq g(x_1, \dots, x_n)$ for all $a_i \in \mathcal{A}$. We also write $f \equiv_{\mathcal{A}} g$.

Although homomorphisms are best described in terms of congruences, as to be expected, these have a special interplay with \mathcal{T} as described in Proposition 8.16. The categorical approach also permits one to bring in standard categorical tools such as tensor products (§8.5).

Our next task is to see what can be said in linear algebra over a meta-tangible system. For example, the main result unifying different notions of matrix rank in [48, Theorem 3.4], is formulated rather transparently in this far more general context (for tangible vectors) in §9, but only one direction holds in general: (Theorem 9.16):

A square matrix A over a meta-tangible system is singular if its rows are dependent.

A wide-ranging counterexample for the other direction is presented in [2], thereby giving a negative answer to a question raised verbally by Baker concerning the ranks of matrices over hyperring. This flavor of the theory seems to depend on the kind of negation map, since we have a positive result for systems of the first kind in [5].

Details of how symmetrization works for matrix theory are given in §9; most notably, for “ $(-)$ -determinants” of matrices (§9.1.1, in analogy to [2]) and classical matrix monoids (§9.1.4).

Having the basic theory in place, we return in §10 to the mainstay of tropical mathematics, which is tropicalization, with an eye towards finding analogous versions of tropicalization as application of the systemic approach. Tropicalization is explained as a \preceq -morphism of systems in §10.1, especially Proposition 10.2. This enables us to define tropical concepts precisely in terms of the tropicalization of the corresponding classical concepts. The advantage of this point of view is to make universal algebra available as a guide. The relevant classes of algebras should be closed under taking subalgebras (in the context of universal algebra). This formulation is followed in §11 to provide compatible definitions and initial investigation of Grassmann algebras (§11.1), Lie semialgebras (§11.2), Lie super-semialgebras (§11.2.1), and Poisson algebras (§11.4). Negation is an important ingredient in these definitions.

2. BACKGROUND FROM UNIVERSAL ALGEBRA

Before implementing the program in detail, we review a few notions from universal algebra, a venerable theory from the early 20th century which was largely superceded by the more general theory of categories, but which is particularly apt when we want to specify algebraic structures. In particular, the structure theory of semirings[†] is motivated by general considerations from universal algebra, for which we use [50] as our reference, also cf. [15], but which we modify slightly in order to deal with more sophisticated algebraic structures.

2.1. Algebraic structures.

Definition 2.1. A **carrier** is a collection of sets $\{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_t\}$. A set of **operators** is a set $\Omega := \cup_{m \in \mathbb{N}} \Omega(m)$ where each $\Omega(m)$ in turn is a set of formal m -ary symbols $\{\omega_{m,j} = \omega_{m,j}(x_{1,j}, \dots, x_{m,j}) : \omega_{m,j} \in \Omega(m), j \in J\}$, interpreted as maps $\omega_{m,j} : \mathcal{A}_{j_1} \times \dots \times \mathcal{A}_{j_m} \rightarrow \mathcal{A}_{i_{m,j}}$. Each operator $\omega_{m,j}$, called an **$(m$ -ary) operator**, has a **target** $\mathcal{A}_{i_{m,j}}$ of **index** $i_{m,j}$, indicating where the operator takes its values. The 0-ary operators are just distinguished elements, that we call **constants**.

We define an **Ω -formula** inductively: Each formal letter $x_{u,i}$ is an Ω -formula with **target** i , and if ϕ_u are Ω -formulas with respective targets i_u , $1 \leq u \leq m$, and if $\omega_{m,j}(x_{1,j}, \dots, x_{m,j}) \in \Omega$ is compatible with ϕ_u in the sense that $i_u = j_u$ for x_{j_u} for each u , then $\omega_{m,j}(\phi_1, \dots, \phi_m)$ also is an Ω -formula. ,

An **identical relation** is a pair (ϕ, ψ) of Ω -formulas (having the same target), for which (ϕ, ψ) satisfies $\phi(a_0, \dots, a_\ell) = \psi(a_0, \dots, a_\ell)$ for all $a_u \in \mathcal{A}_{j_u}$ for each carrier $\{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_t\}$.

A **signature** is a pair (Ω, \mathcal{I}) , where Ω is a set of operators and \mathcal{I} is a set of identical relations. A signature is **additive** if it contains the operator of addition, satisfying the associative identity, which will be treated differently from the other operators.

Writing \mathcal{I} for the set of identical relations, we also call the carrier $\{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_t\}$ an **$(\Omega; \mathcal{I})$ -algebra**, otherwise known in the literature as an **algebraic structure**. In summary, each carrier is a collection of sets endowed with operators from Ω , which satisfies each identical relation.

In this paper, all of our signatures are additive. When we want to stress the special role of addition, we repeat the designation “additive.” On the other hand, when our structure has multiplication (such as when it is cancellative), we consider multiplication as just one more operator in the signature.

In order to avoid cumbersome notation, we denote the $(\Omega; \mathcal{I})$ -algebra $\{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_t\}$ just as \mathcal{A} for convenience, and often are referring principally to the target, where the other \mathcal{A}_j are “secondary.” Often \mathcal{A} just denotes the “main” structure.

Example 2.2.

- (i) \mathcal{A}_1 is a semiring[†] R , and \mathcal{A}_2 is an R -module.
- (ii) There is another way of thinking of (i). Instead of dealing with \mathcal{A}_1 separately, one only has \mathcal{A}_2 , and for each $r \in \mathcal{A}_1$ one defines the 1-ary operator $\omega_r : \mathcal{A}_2 \rightarrow \mathcal{A}_2$ via $\omega_r(a) = ra$. Distributivity could be expressed as the identical relations given by $\omega_{r_1+r_2}(x) = \omega_{r_1}(x) + \omega_{r_2}(x)$ and $\omega_r(x_1+x_2) = \omega_r(x_1) + \omega_r(x_2)$. Associativity could be handled similarly.
- (iii) \mathcal{A}_2 is a module over a semifield $\mathcal{A}_1 = F$, and our set of operators includes both a bilinear form $\mathcal{A}_2 \times \mathcal{A}_2 \rightarrow F$ and a quadratic form $\mathcal{A}_2 \rightarrow F$.
- (iv) C -semialgebras (Definition 1.5), where $\mathcal{A}_1 = C$.

Example 2.3. Our major example, the system, involves set inclusion $(\mathcal{T} \subseteq \mathcal{A})$, which is not expressed directly in universal algebra, so the signature requires some care in this case. We have $\mathcal{A}_1 = \mathcal{A}$ and $\mathcal{A}_2 = \mathcal{T}$, where we have a unary operation $\mu : \mathcal{T} \rightarrow \mathcal{A}$. \mathcal{A} itself is a semigroup with respect to a binary operator $+$, which is not closed on \mathcal{T} . But every other operator ω in the signature of \mathcal{A} is also required to be defined on \mathcal{T} and commutes with μ in the sense that the identical relation

$$\mu(\omega(x_1, \dots, x_m)) = \omega(\mu(x_1), \dots, \mu(x_m))$$

is satisfied where $x_i \in \mathcal{T}$. In particular the negation map, a unary operator on \mathcal{A} , does restrict to a unary operator on \mathcal{T} . Likewise, \mathcal{T} may have a distinguished element $1_{\mathcal{T}}$ whose image in \mathcal{A} is the distinguished element $1_{\mathcal{A}}$. Now, passing to $\mu(\mathcal{T})$ enables us to envision $\mathcal{T} \subseteq \mathcal{A}$. In other words, when we write \mathcal{T} we really mean $\mu(\mathcal{T})$. We do not require $0_{\mathcal{A}}$ to be in \mathcal{T} , but it acts as the additive identity element on the elements of \mathcal{T} .

\mathcal{A} itself is stipulated to be a semigroup with respect to a binary operator $+$, which is not closed on \mathcal{T} . But the negation map, a unary operator on \mathcal{A} does restrict to a unary operator on \mathcal{T} .

Often \mathcal{A} itself is a semiring[†], and \mathcal{T} a multiplicative submonoid, even a group. If we also want \mathcal{A} to be a semialgebra say over a commutative semiring[†] R , we could take \mathcal{A} to be an Abelian semigroup, together with a “scalar multiplication” $R \times \mathcal{A} \rightarrow \mathcal{A}$ satisfying the usual bilinear relations as well as $(r_1 r_2)a = r_1(r_2 a) = (r_1 a)r_2$.

The concept of \mathcal{T} -semiring (Definition 2.13 below) also is easily defined in these terms.

2.1.1. Constants.

Remark 2.4. The constants in our signature are the 0-ary operators, and all Ω -formulas (and thus all identical relations) can be thought of as algebraic expressions whose coefficients are constants.

This leads us to the basic question, “What are the explicit constants in our signature?” Of course this depends on the particular signature, but often one starts with 1 , the multiplicative unit when it exists.

The other constants of \mathcal{A} come from addition, so we do not require them in \mathcal{T} . Any constant c gives rise to the constant $c^\circ := c(-)c$. One puts in e, e' from Equation (1.5).

Likewise we have $\mathbf{1} = 1$, $\mathbf{2} = 1 + 1$, etc. from Definition 1.20, and one could then define more constants (such as $\sqrt{3}$) by means of algebraic equations. In this way, we could for example designate constants of \mathbb{Q}_{\max} . But we cannot obtain any more constants from the max-plus algebra \mathbb{R}_{\max} , since there is no way to specify irrational numbers!

2.2. Varieties.

The class of $(\Omega; \mathcal{I})$ -algebras of a given signature comprise a category. Given two $(\Omega; \mathcal{I})$ -algebras \mathcal{A} and \mathcal{B} , one defines an Ω -homomorphism to be maps $\varphi_u : \mathcal{A}_u \rightarrow \mathcal{B}_u$, preserving the operators in the obvious way:

$$\varphi_{i_j}(\omega_{m,j}(a_1, \dots, a_m)) = \omega_{m,j}(\varphi_{j_1}(a_1), \dots, \varphi_{j_1}(a_m)), \quad \forall a_k \in \mathcal{A}_{j_k}. \quad (2.1)$$

Thus, we have a category \mathcal{C} whose objects are the $(\Omega; \mathcal{I})$ -algebras and whose morphisms are the Ω -homomorphisms. (But we will utilize our surpassing relation to modify (2.1) in defining morphisms, as hinted in Example 1.41(iii).) Since the indices are understood but complicate the notation, we will denote a Ω -homomorphism merely as φ .

Definition 2.5. An $(\Omega; \mathcal{I})$ -*subalgebra* of an $(\Omega; \mathcal{I})$ -algebra $\mathcal{A} = \{\mathcal{A}_1, \dots, \mathcal{A}_t\}$ is a collection of subsets $\{\mathcal{A}'_1, \dots, \mathcal{A}'_t\}$ closed under the operators $\omega_{m,j}$; it can also be viewed categorically as an equivalence class of monic Ω -homomorphisms into \mathcal{A} .

Lemma 2.6. The class of $(\Omega; \mathcal{I})$ -algebras is an **algebraic variety**, in the following sense:

- (i) Any $(\Omega; \mathcal{I})$ -subalgebra of an $(\Omega; \mathcal{I})$ -algebra is itself an $(\Omega; \mathcal{I})$ -algebra;
- (ii) If $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ is an Ω -homomorphism, then $\varphi(\mathcal{A})$ also is a $(\Omega; \mathcal{I})$ -subalgebra of \mathcal{B} ;
- (iii) The Cartesian product $\prod_{i \in I} \mathcal{A}_i$ of $(\Omega; \mathcal{I})$ -algebras is an $(\Omega; \mathcal{I})$ -algebra under the componentwise operations.

Proof. (i) The identical relations hold a fortiori.

(ii) The identical relations clearly hold under homomorphic images, and so yield a subalgebra by Definition 2.5.

(iii) Given $\omega_{m,j} \in \Omega$ and $(a_{i,k}) \in \prod \mathcal{A}_{i_{jm}}$ for $1 \leq k \leq m$, we define

$$\omega_{m,j}((a_{i,1}), \dots, (a_{i,m})) = (\omega_{m,j}(a_{i,1}, \dots, a_{i,m})).$$

□

2.3. Partial orders in universal algebra.

Let us see how partial orders fit into the language of universal algebra. We have a natural generalization of ordered monoid.

Definition 2.7. A (Ω, \mathcal{I}) -**PO** is a $PO \leq$ that respects the various operators; i.e., for each $\omega_{m,j} \in \Omega$, if $a_{1,k} \leq a_{2,k}$ for $1 \leq k \leq m$, then

$$\omega_{m,j}(a_{1,1}, \dots, a_{1,m}) \leq \omega_{m,j}(a_{2,1}, \dots, a_{2,m}).$$

Unless otherwise indicated, “partial order” refers to an (Ω, \mathcal{I}) -partial order (with (Ω, \mathcal{I}) understood). When \leq is not equality, this definition restricts the signatures at our disposal. For example, the usual ordered fields such as \mathbb{R} and \mathbb{Q} do not preserving the order under multiplication; $-1 > -2$ but $(-1)^2 < (-2)^2$. In fact, we have the following fact.

Lemma 2.8. Suppose (Ω, \mathcal{I}) contains the unary operation of inverse on a multiplicative group \mathcal{T} . Then any (Ω, \mathcal{I}) -partial order \leq restricts to equality on \mathcal{T} .

Proof. Suppose $a_1 \leq a_2$. By definition $a_1^{-1} \leq a_2^{-1}$, so

$$a_2 = a_2 a_1^{-1} a_1 \leq a_2 a_2^{-1} a_1 = a_1 \leq a_2,$$

so equality holds at each stage.

□

Lemma 2.9.

- (i) Any partial order \leq on an $(\Omega; \mathcal{I})$ -algebra \mathcal{A} induces a partial order on each $(\Omega; \mathcal{I})$ -subalgebra.
- (ii) Any partial order \leq on $(\Omega; \mathcal{I})$ -algebras \mathcal{A}_i induces a partial order on $\prod_i \mathcal{A}_i$, via $(a_i) \leq (b_i)$ iff $a_i \leq b_i$ for each i .

Proof. (i) holds a fortiori.

(ii) By components.

□

On the other hand, a homomorphic image of an ordered semiring[†] need not be ordered. For example, \mathbb{Z}_2 is a homomorphic image of \mathbb{Z} .

2.4. (Optional) Application: \mathcal{T} -semirings[†] and their modules.

This subsection is included for those readers who would like to see later on how hypergroups fit into the theory. The motivation grew out of a conversation with Baker, in which we realized that the “tropical hyperfield” of [7] and [74, §5.2] is isomorphic to the “extended” tropical arithmetic in Izhakian’s Ph.D. dissertation (Tel-Aviv University) in 2005, also cf. [38], and given more formally in [47]. Thus one would like to see how other major hyperrings also can be studied by the more amenable semiring theory (with some modification concerning distributivity), which fits in well with the general theory of universal algebra. The reader not interested in hypergroups could skip this discussion, and read “semiring” for “ \mathcal{T} -semiring,” but would miss some nice examples which have motivated many of our results.

To include applications to hypergroups, we need to weaken the notion of semiring.

Definition 2.10. A *pre-semiring*[†] is a set $(R, +, \cdot, \mathbb{1}_R)$ for which $(R, +)$ is an additive Abelian semigroup and $(R, \cdot, \mathbb{1}_R)$ is a multiplicative monoid, but not necessarily satisfying the usual distributive laws.

A *pre-semiring* is a pre-semiring[†] with an absorbing zero element.

Since associativity of matrix multiplication depends on distributivity on R , $M_n(R)$ may fail to be a pre-semiring even when R is, but that does not deter us from defining the determinant and the matrix adjoint in the usual way.

Clearly the definitions of negation map and \mathcal{T} -surpassing relation can be repeated for pre-semirings, since they do not require distributivity. We also can put \mathcal{T} -semirings[†] (perhaps with a negation map) in the universal algebraic context, by taking identical relations which say that multiplication by elements of \mathcal{T} distributes over addition of the other elements of \mathcal{A} .

Thus, we can define triples and systems, and adapt the program of §5 for pre-semirings. We do slightly modify the definition of module.

Definition 2.11. Given a pre-semiring[†] R with submonoid \mathcal{T} and with the surpassing relation \preceq , a *\mathcal{T} -premodule with negation* is an additive semigroup $(M, +, \mathbb{0}_M)$ with a negation map, together with scalar multiplication $R \times M \rightarrow M$ and the corresponding surpassing relation satisfying the following axioms, $\forall t, u \in \mathbb{N}$, $r, r_i \in \mathcal{T}$, $a, a_i \in M$:

- (i) $(-)(ra) = r(-)a$,
- (ii) $(\sum_{i=1}^t r_i)(\sum_{j=1}^u a_j) \preceq \sum_{i=1}^t \sum_{j=1}^u (r_i a_j)$,
- (iii) $(r_1 r_2)a = r_1(r_2 a)$,
- (iv) $\mathbf{n}a = \sum_{i=1}^n a$ for all $a \in \mathcal{T}$,
- (v) $r\mathbb{0}_M = \mathbb{0}_M$.

In the framework of universal algebra, we define the left multiplication maps $\ell_r : M \rightarrow M$ by $\ell_r(a) = ra$, a unary operator for each $r \in \mathcal{T}$, and rewrite these axioms as identical relations.

Proposition 2.12. Suppose $(M, +)$ is a semigroup with a surpassing relation \preceq , and \mathcal{T} is a multiplicative group, with a multiplication $\mathcal{T} \times M \rightarrow M$ satisfying $r\mathbb{0}_M = \mathbb{0}_M$. If the left multiplication maps $\ell_r : M \rightarrow M$ are morphisms for each $r \in \mathcal{T}$, then M is a \mathcal{T} -premodule.

Proof. We are given

- (i) $r(\sum_{i=1}^t a_i) \preceq \sum_{i=1}^t (ra_i)$, for all r in \mathcal{T} , $a_i \in M$.
- (ii) $(r_1 r_2)a \preceq r_1(r_2 a)$,

and we need to prove the conditions of Definition 2.11. First we have (ii) from a standard argument given below in Lemma 8.8. To see (i),

$$\sum_{i=1}^t (ra_i) = (rr^{-1}) \sum_{i=1}^t (ra_i) = r(r^{-1} \sum_{i=1}^t (ra_i)) \preceq r \sum_{i=1}^t (r^{-1}(ra_i)) = r \sum_{i=1}^t ((r^{-1}r)a_i) = r \sum_{i=1}^t a_i.$$

□

Definition 2.13. When $M = R \cup \{\mathbb{0}_R\}$; then we call M a *\mathcal{T} -semiring*.

In other words, a \mathcal{T} -semiring is a pre-semiring R with $\mathcal{T} \subseteq R$, for which distributivity is required only with respect to sums of elements of \mathcal{T} .

2.5. Congruences.

Ideals do not work well for universal algebras without negatives, but fortunately they have a replacement in universal algebra. We recall as a special case from [50, p. 61] that a **congruence** Φ on a carrier \mathcal{A} is an equivalence relation \equiv preserving the operators of the signature. For us this will include addition and usually multiplication, i.e., if $a_i \equiv b_i$ then $a_1 + a_2 \equiv b_1 + b_2$ and $a_1 a_2 \equiv b_1 b_2$. Sometimes we denote Φ as the relation \equiv , or, equivalently, as $\{(a, b) : a \equiv b\}$, a subalgebra of $\mathcal{A} \times \mathcal{A}$.

Just as ideals arise in the classical algebraic structure theory, congruences play the analogous role in universal algebra. Any semiring[†] homomorphism $\psi : \mathcal{A} \rightarrow \mathcal{A}'$ gives rise to a congruence Φ_ψ on \mathcal{A} given by $(a, b) \in \Phi_\psi$ iff $\psi(a) = \psi(b)$; conversely, any congruence Φ gives rise to a carrier \mathcal{A}/Φ on the equivalence classes, and a natural Ω -homomorphism $\psi : \mathcal{A} \rightarrow \mathcal{A}/\Phi$ given by $a \mapsto [a]$.

Whereas for R a ring, any submodule N of an R -module M defines a congruence Φ given by $a \equiv b$ iff $a - b \in N$, this is not relevant to semirings[†], which is why we need to pass to congruences.

Definition 2.14. *If Φ is a congruence, we define the **factor algebra***

$$\mathcal{A}/\Phi = \{[a] : a \in \mathcal{A}\},$$

where the equivalence classes are taken with respect to Φ .

Example 2.15. *Define a congruence Φ by the equivalence $a \equiv b$ iff $a^\circ = b^\circ$. Passing to \mathcal{A}/Φ eliminates the classical part of \mathcal{T} .*

*If $\Phi_1 \subseteq \Phi_2$ are congruences, we define the **factor congruence***

$$\Phi_1/\Phi_2 = \{([a], [b]) : (a, b) \in \Phi_1\},$$

where the equivalence classes are taken with respect to Φ_2 . Of course Φ_1/Diag_A is just Φ_1 .

A **congruence homomorphism** is a homomorphism of congruences, viewed as $\mathcal{A} \times \mathcal{A}$ -subalgebras. (Here \mathcal{A} usually will be a module, in which context “subalgebra” means submodule.)

Lemma 2.16. *For congruences $\Phi_1 \subseteq \Phi_2$, there is a congruence homomorphism $\Phi_1 \rightarrow \Phi_1/\Phi_2$ given by*

$$(a, b) \mapsto ([a], [b]).$$

Proof. $[a] = [b]$ iff $(a, b) \in \Phi_2$. □

We also want to view images as congruences.

Lemma 2.17. *Given any congruence homomorphism $\hat{f} : \Phi \rightarrow \Phi'$, the set $\hat{f}(\Phi) := \{\hat{f}(a, b) : (a, b) \in \Phi\}$ is a congruence contained in Φ' .*

Proof. By definition of congruence homomorphism, $\hat{f}(\Phi)$ is closed under all of the operators. □

2.6. Free algebras.

We recall the basic construction of free objects in categories, from [15].

Construction 2.18. *Given a signature $(\Omega; \mathcal{I})$ and an index set J , there is a well-known free $(\Omega; \mathcal{I})$ -algebra $\mathcal{F}_{(\Omega; \mathcal{I})}$, in the sense that for any $(\Omega; \mathcal{I})$ -algebra \mathcal{A} and any set $\{a_j : j \in J\} \subseteq \mathcal{A}$, there is a unique homomorphism $F_{(\Omega; \mathcal{I})} \rightarrow \mathcal{A}$ sending each $x_j \mapsto a_j$. $F_{(\Omega; \mathcal{I})} \rightarrow \mathcal{A}$ is constructed in two steps: First we take the case where $\mathcal{I} = \emptyset$, with $\mathcal{F}_{(\Omega; \emptyset)}$ obtained by starting with indeterminates x_j , and continuing inductively, taking an indeterminate $x_{\phi(x_1, \dots, x_m)}$ for each formula ϕ . Clearly for any $(\Omega; \mathcal{I})$ -algebra \mathcal{A} and any $\{a_j : j \in J\} \subseteq \mathcal{A}$, there is a unique Ω -homomorphism $\varphi : \mathcal{F}_{(\Omega; \emptyset)} \rightarrow \mathcal{A}$ satisfying $x_j \mapsto a_j$, for all j in J .*

Given a general set \mathcal{I} of universal identities, we define an equivalence relation on $\mathcal{F}_{(\Omega; \emptyset)}$, by stipulating that $f(x_1, \dots, x_\ell) \equiv g(x_1, \dots, x_\ell)$, iff the sentence “ $f(x_1, \dots, x_\ell) = g(x_1, \dots, x_\ell)$ ” is in \mathcal{I} . This is a congruence, so the set of equivalence classes constitutes an algebra $F_{(\Omega; \mathcal{I})} = F_{(\Omega; \emptyset)}/\Phi$ of signature (Ω, \mathcal{I}) . We write \bar{x}_j for the equivalence class of x_j .

Lemma 2.19. *$F_{(\Omega; \mathcal{I})}$ is the free $(\Omega; \mathcal{I})$ -algebra.*

Proof. Suppose \mathcal{A} is an $(\Omega; \mathcal{I})$ -algebra. Take the unique Ω -homomorphism $\varphi : \mathcal{F}_{(\Omega; \emptyset)} \rightarrow \mathcal{A}$ satisfying $x_i \mapsto a_i$, for all i in I . Taking the congruence $\Phi = \mathcal{I}$, we see by definition that φ factors through Φ , so induces an Ω -homomorphism $\bar{\varphi} : \mathcal{F}_{(\Omega; \emptyset)} / \Phi \rightarrow \mathcal{A}$, given by sending \bar{x}_j to a_j . φ factors uniquely through \mathcal{I} , since otherwise one could lift back to a different Ω -homomorphism $C\{x; \Omega\} \rightarrow \mathcal{A}$. Hence is the free $(\Omega; \mathcal{I})$ -algebra. \square

Since this construction is completely formal, we have no knowledge even of whether the free $(\Omega; \mathcal{I})$ -algebra is trivial or not, but there are standard constructions such as the free module, the free semiring, and the free algebra, that we will utilize.

2.6.1. The free semiring[†] and semialgebra.

We recall some common instances of this general construction. We already noted the free module in Definition 1.3.

Definition 2.20. The **free monoid** \mathcal{M} is the monoid in formal indeterminates, with multiplication given by concatenation.

The **free semiring**[†] then is $\mathbb{N}[\mathcal{M}]$ where \mathcal{M} is the free monoid.

Example 2.21. The free \mathbb{N}_{\max} -semialgebra is the monoid semialgebra $\mathbb{N}_{\max}[\mathcal{M}] = \mathbb{N}_1[\mathcal{M}]$, taken in the context of Remark 1.57. In other words, $\bar{n}x$ evaluates as x in $\mathbb{N}_{\max}[\mathcal{M}]$.

The construction of the **free pre-semiring**[†] requires more subtlety, since we do not have distributivity at our disposal. We can define in accordance with the universal algebraic approach of Construction 2.18, by taking a new indeterminate for each formal expression (viewing the negation map as a unary operation) in the previously defined words. (We get back to the previous constructions via distributivity where applicable. To obtain distributivity over \mathcal{T} we can make these reductions over the elements of \mathcal{T} .)

2.7. Homogeneous and multilinear operators.

We need some technical conditions to work with operators. We assume that our signature is additive. Our guiding principle is that our operators will not get in the way of the theory. Accordingly, we call an operator ω_m **\mathcal{T} -regular** if $\omega_m(a_1, \dots, a_k, \dots, a_m) \in \mathcal{T}$ for all $a_i \in \mathcal{T}$. For example, if \mathcal{T} is a monoid, then multiplication is regular, but addition is never regular since $a(-)a \notin \mathcal{T}$. A triple is **\mathcal{T} -regular** if all of its operators other than addition are regular.

2.7.1. Linearization of additive signatures.

Definition 2.22. An operator ω_m is **k -linear** if

$$\omega_m(a_1, \dots, a_k + a'_k, \dots, a_m) = \omega_m(a_1, \dots, a_k, \dots, a_m) + \omega_m(a_1, \dots, a'_k, \dots, a_m)$$

for all $a_k, a'_k \in \mathcal{A}_{j_k}$.

When the signature contains an action $\mathcal{T} \times \mathcal{A}_{j_k} \rightarrow \mathcal{A}_{j_k}$, an operator ω_m is **k -homogeneous of degree d** if

$$\omega_m(a_1, \dots, ra_k, \dots, a_m) = r^d \omega_m(a_1, \dots, a_k, \dots, a_m)$$

for all $r \in \mathcal{T}$ and $a_k \in \mathcal{A}_{j_k}$.

When the signature contains the structure of R -modules, an operator ω_m is **R -multilinear** if k -linear and k -homogeneous of degree 1, for each $1 \leq k \leq m$.

Note that addition itself is not multilinear, since $(a_0 + a_2) + a_3 \neq (a_0 + a_3) + (a_2 + a_3)$ in general, and likewise the multiplicative inverse is not multilinear, but except for that, we only work with operators that are k -homogeneous for each k .

Definition 2.23. Given an operator $\omega = \omega(x_1, \dots, x_m)$, we define its **j -partial linearization** to be homogeneous operators $\omega'_{j;d}(x_1, \dots, x_j, x'_j, \dots, x_m)$ of degree d such that

$$\omega(x_1, \dots, x_j + x'_j, \dots, x_m) = \omega(x_1, \dots, x_j, \dots, x_m) + \omega(x_1, \dots, x'_j, \dots, x_m) + \sum_d \omega'_{j;d}(x_1, \dots, x_j, x'_j, x_m).$$

The operator ω is **multilinearizable** (resp. **R -multilinearizable**) if one can apply partial linearization a finite number of times to get to multilinear (resp. R -multilinear) operators.

Our signatures will often be closed under this linearization process. The choice of ω'_j is far from unique. For example, if we take a quadratic form Q , its underlying bilinear form would be Q'_1 , so Q is linearizable; the extent of the non-uniqueness of the underlying bilinear form of a quadratic form is one of the main concerns of [43, 44].

Definition 2.24. An *ideal* of an $(\Omega; \mathcal{I})$ -algebra \mathcal{A} is an $(\Omega; \mathcal{I})$ -subalgebra \mathcal{I} satisfying

$$\omega_m(a_1, \dots, b, \dots, a_m) \in \mathcal{I}$$

for all $a_k \in \mathcal{A}$ and $b \in \mathcal{I}$.

In particular, a *ideal* of a pre-semiring \mathcal{A} is an additive semigroup I such that $aI, Ia \subseteq I$ for all $a \in \mathcal{A}$. (In other words, it is an ideal in the sense of universal algebra.)

$\{0\}$ is an ideal, but one of the main ideas promoted here is that it can be replaced in the theory by other ideals.

Lemma 2.25. If the signature is closed under linearization, then, for any ideal \mathcal{I} ,

$$\omega_m(a_1, \dots, a_j + b, \dots, a_m) \in \omega_m(a_1, \dots, a_j, \dots, a_m) + \mathcal{I}$$

for each $b \in \mathcal{I}$.

Proof. $\omega_m(a_1, \dots, a_j + b, \dots, a_m) = \omega_m(a_1, \dots, a_j, \dots, a_m) + \text{terms in } \mathcal{I}$. \square

3. TROPICAL EXAMPLES VIEWED IN TERMS OF UNIVERSAL ALGEBRA

Let us pause to see how well the familiar structures of tropical mathematics fit into the theory of universal algebra and varieties.

3.1. Varieties arising naturally in tropical mathematics.

Example 3.1.

- (i) **Semirings[†].** Since tropical algebra originates with the max-plus algebra, our most basic example is the variety of semirings[†] $(R, \mathbb{1}_R, +, \cdot)$, in which each $\mathcal{A}_j = R$, which has the constant $\mathbb{1}_R$, the multiplicative unit, and the binary operators $+$ of addition and \cdot of product. The identical relations are given by $x \cdot \mathbb{1}_R = x$, $\mathbb{1}_R \cdot x = x$, $x_1 + x_2 = x_2 + x_1$, $(x_1 + x_2) + x_3 = x_1 + (x_2 + x_3)$, and distributivity. For associative semirings[†], we impose $(x_1 x_2) x_3 = x_1 (x_2 x_3)$.
For commutative semirings[†], we impose commutativity of multiplication. Not having to deal with 0 , we can also describe semifields[†] directly in terms of the extra unary operator $^{-1}$, together with the identical relation $xx^{-1} = \mathbb{1}$.
- (ii) **Semirings.** We include a constant 0_R for the additive identity, (and its accompany identical relation $x + 0_R = x$), to the signature, to define the variety of semirings.
- (iii) **Rings.** Rings are defined precisely as in (ii), but also with an extra 1-ary operator $\omega_{1,0} : \mathcal{A} \rightarrow \mathcal{A}$ for negation $a \mapsto -a$, together with the identical relation $a + (-a) = 0$. (The lack of negation in semirings is what motivates this paper.)
- (iv) **Idempotent semirings[†].** This can be written easily as an identical relation, but “bipotency” cannot, as to be discussed in Example 3.3(i).
- (v) **Modules over semirings[†].** One can also work with modules over semirings[†], by designating the semiring[†] R and its module M as \mathcal{A}_1 and \mathcal{A}_2 respectively, together with the constant 0_M and the binary operator $R \times M \rightarrow M$ satisfying the usual module axioms, written as identical relations, including the identical relation $x0_M = 0_M$.
- (vi) **Semialgebras.** One just combines the semiring[†] and module axioms, as well as (1.1), written as identical relations.
- (vii) **ν -semirings[†].** One can describe ν -semirings[†] (Definition 1.24) in the context of universal algebra, by declaring the constant $e := 0^\nu$ to be both an additive and multiplicative idempotent, i.e., $e + e = e$ and $e^2 = e$. Then $re = r^\nu$, so the map $r \mapsto re$ is the operator corresponding to the ghost map. This is explained in [41, Remark 2.1]. But supertropical semirings[†] are out of this scope, cf. Example 3.3(ii).

- (viii) **Matrix semirings.** Matrices can be defined over any semiring R ; one defines matrix addition and multiplication in the usual way. The $n \times n$ matrix structure can be obtained as in [67, Proposition 13.9] in terms of matrix units $\{e_{ij} : 1 \leq i, j \leq n\}$ viewed as constants satisfying the universal relations

$$\sum_{i=1}^n e_{ii} = \mathbb{1}_R; \quad e_{ij}e_{kl} = \delta_{j,k}e_{il}, \quad 1 \leq i, j, k, \ell \leq n.$$

The point is that the standard proof given in [67, Proposition 13.9] does not use negation. Matrices gives rise to the trace operator $\text{tr}(r) = \sum e_{ii}re_{ii}$. Note that over the supertropical semiring[†], for $n \geq 2$, that $\text{tr}(I) = \mathbb{1}_R^\nu$. The determinant is more problematic since the classical formula involves minus signs; we will return later to this issue.

- (ix) **Formal traces.** Since much of linear algebra involves the trace bilinear form, let us formalize the trace from the previous example and define a trace operator $\text{tr} : \mathcal{A} \rightarrow R$ satisfying the identical relations

$$\text{tr}(x_1x_2) = \text{tr}(x_2x_1); \quad \text{tr}(\mathbb{1}_{\mathcal{A}}) = \mathbb{1}_R^\nu.$$

(Strictly speaking, this axiomatic formulation is for degree $n \geq 2$.)

- (x) **Bilinear forms.** We start with the set-up of (v), but now also have an operator $b : M \times M \rightarrow R$ satisfying the identical relations defining bilinearity.
- (xi) **Quadratic forms.** The general definition of quadratic form over a semiring is given in [43]. Continuing (x), we introduce a (quadratic) operator $Q : M \rightarrow R$ satisfying the identical relation $Q(x+y) = Q(x) + Q(y) + b(x, y)$, where $b(x, y)$ is a bilinear form. Note that $b(x, y)$ is a linearization of Q .
- (xii) **Blueprints.** Lorscheid [59, Definition 1.1] has put tropical geometry in a rather general framework, which we review.

Definition 3.2. A **blueprint** B is a monoid A with zero, together with an equivalence relation Φ on the monoid semiring[†] $\mathbb{N}[A] = \{\sum a_i : a_i \in A\}$ (of finite formal sums of elements of A) that satisfies the following axioms (where we write $\sum a_i \equiv \sum a'_j$ whenever $(\sum a_i, \sum a'_j) \in \Phi$:

- (a) The relation Φ is additive and multiplicative. (Thus Φ is a congruence.)
- (b) The absorbing element 0 of A is compatible with the zero of $\mathbb{N}[A]$; i.e., $0 \equiv \text{empty sum}$.
- (c) If $a, b \in A$ and $a \equiv b$, then $a = b$ (as elements in A).

A **homomorphism** $f : B_1 \rightarrow B_2$ of blueprints is a multiplicative map $f : A_1 \rightarrow A_2$ between the underlying monoids of B_1 and B_2 with $f(0) = 0$ and $f(\mathbb{1}) = \mathbb{1}$, such that for every relation $\sum a_i \equiv \sum a'_j$ for B_1 , we have $\sum f(a_i) \equiv \sum f(a'_j)$ for B_2 .

This definition is suited to universal algebra. Namely, the semiring[†] $\mathbb{N}[A]$ can be viewed as the monoid semiring[†] over \mathbb{N} , modulo a congruence.

- (xiii) **Hyperstructures.** Hyperstructures do not fit directly into the language of universal algebra, since the sum of elements is a set, not necessarily an element, but universal algebra is applicable when we pass to the power set, since its elements are subsets; cf. §1.5.

3.2. Structures of tropical mathematics which do not comprise varieties.

There also are several important concepts which fail to correspond to varieties, because one of the key ingredients of Lemma 2.6 is missing, either homomorphic images or direct products, as we see now.

Example 3.3.

- (i) **Ordered semirings[†] versus bipotence.** If one tries to internalize an order into the given binary operator of (bipotent) addition, i.e., putting $a + b = \max\{a, b\}$, one could define the relations

$$x_1 + x_2 = x_1 \quad \vee \quad x_1 + x_2 = x_2.$$

This relation passes to subalgebras and homomorphic images, but not to direct products, since (componentwise) $(1, 2) + (2, 1) = (2, 2)$.

- (ii) **Supertropical semirings[†].** Supertropicality passes to subalgebras and homomorphic images, but not to direct products, just as in (i).

(iii) **ub semigroups.** Any ub semigroup satisfies the sentence

$$x_1 + x_2 + x_3 = x_1 \quad \Rightarrow \quad x_1 + x_2 = x_1.$$

But again this is not a identical relation; it passes to sub-semigroups and direct products, but not to homomorphic images, with the example of $\mathbb{N} \rightarrow \mathbb{Z}_2$, taken modulo 2. (This sentence is an example of what in mathematical logic is known as a “quasi-identity.”)

4. NEGATION MAPS AND SURPASSING RELATION

We are ready for the main theme of this paper, arising from [2], which enables us to treat tropical structures in a way parallel to classical theory. Essentially we are following [27, Definiton 4.1], although here the focus often is on the semigroup structure.

4.1. Negation maps in universal algebra.

Let us incorporate negation maps into universal algebra. We continue from Definition 1.9. We define the negation map as a 1-ary operator $a \mapsto (-)a$, which we adjoin to our additive signature.

Inductively, we write $(-)^1 a = (-)a$, and $(-)^{k+1} a = (-)((-)^k a)$.

Definition 4.1. An Ω -negation map $(-) : \mathcal{A} \rightarrow \mathcal{A}$ satisfies the following:

- Definition 1.9;
- Equation (1.3) or Equation (1.4) when relevant, i.e., when \mathcal{A} is a module or a semiring[†];
- The identical relations corresponding to

$$(-)^d \omega_{m,j}(a_1, \dots, a_{k-1}, a_k, a_{k+1}, \dots, a_m) = \omega_{m,j}(a_1, \dots, a_{k-1}, (-)a_k, a_{k+1}, \dots, a_m) \quad (4.1)$$

on k -homogeneous operators $\omega_{m,j}$ of degree d (other than addition), for each $a_k \in \mathcal{A}_k$, $1 \leq k \leq m$, $\forall \omega \in \Omega$.

The negation map itself is incorporated into the signature, as a unary operator.

In particular we have the variety of R -modules with a negation map, a key ingredient of a categorical approach to be pursued later.

Intuitively, Definition 1.3 means that the negation map is preserved by each operator other than addition, but $-a + b \neq -(a + b)$. If our additive signature contains a designated submonoid \mathcal{T} , we also assume that $(-)a = ((-)\mathbb{1})a \in \mathcal{T}$ for each $a \in \mathcal{T}$.

Remark 4.2. $((-)\mathbb{1})^2 = (-)((-)\mathbb{1}) = (-)((-)\mathbb{1}) = \mathbb{1}$.

Lemma 4.3. If \mathcal{A} has a negation map, then for any congruence Φ on \mathcal{A} , the induced map

$$(-)(a_0, a_1) = ((-)\mathbb{1})a_0, ((-)\mathbb{1})a_1$$

is a negation map on Φ .

Proof. Check at each component. □

4.1.1. Negation maps on modules.

We need the free module in this context.

Example 4.4. The **free module with negation map** is the free module whose base is formally denoted as $\{e_i, (-)e_i : i \in I\}$, with negation map given by $e_i \mapsto (-)e_i$ and $(-)e_i \mapsto e_i$. In other words,

$$(-) \left(\sum (\alpha_i e_i (-) \beta_i e_i) \right) = \sum (\beta_i e_i (-) \alpha_i e_i).$$

Definition 4.5. If a module M has a negation map $(-)$, we require a congruence Φ on $(M, (-))$ to be closed under negation, i.e., if $(a_0, a_1) \in \Phi$, then $((-)\mathbb{1})a_0, ((-)\mathbb{1})a_1 \in \Phi$.

Lemma 4.6. Given any element $\mathbb{1}'_R$ whose square is $\mathbb{1}_R$, we can define a negation map $(-)$ on R given by $a \mapsto \mathbb{1}'_R a$.

Proof. $(-)(ab) = \mathbb{1}'_R(ab) = (\mathbb{1}'_R a)b = ((-)a)b$, $(-)ab = (\mathbb{1}'_R)ab = a(\mathbb{1}'_R b) = a((-)b)$, and

$$(-)((-)a) = \mathbb{1}'_R(\mathbb{1}'_R a) = \mathbb{1}'_R{}^2 a = a.$$

$$\omega(a_1, \dots, (-)a_k, \dots, a_m) = (\mathbb{1}'_R)^{d_k} \omega(a_1, \dots, a_k, \dots, a_m) = (-)^{d_k} \omega(a_1, \dots, a_k, \dots, a_m).$$

□

In particular, $(-)\mathbb{1} = \mathbb{1}'_R$.

Of course in the max-plus algebra we do not have nontrivial negation maps, which is why we will need to symmetrize the structure to adjoin $(-)\mathbb{1}_R$. Also a module M may have a negation map even if the underlying semiring[†] R lacks a negation map, for example in the case of (tropical) Grassmann and Lie semialgebras, to be discussed.

4.1.2. Blueprints with a negation map.

Following the notation of [59], we have:

Lemma 4.7. *Suppose a monoid A has a given negation map $(-)$. Then any A -blueprint B has the negation map given by $(-)a = ((-)\mathbb{1})a$.*

Proof. We verify the extra relation:

$$\text{If } \sum a_i \equiv \sum b_j, \text{ then } \sum (-)a_i = \sum (-)\mathbb{1}a_i \equiv (-)\mathbb{1} \sum a_i \equiv (-)\mathbb{1} \sum b_j \equiv \sum (-)b_j.$$

□

4.2. Combining and comparing negation maps.

Suppose that we have several negation maps.

Lemma 4.8. *Any two negation maps commute.*

Proof. This is because the negation maps are in the signature. Thus, if $(-), (-)'$ are negation maps, we have

$$(-)((-)'a) = (-)'((-)a)$$

for all a .

□

Proposition 4.9. *The composite of two negation maps with the same surpassing relation is a negation map which preserves this surpassing relation.*

Proof. If $(-), (-)'$ are commuting negation maps on \mathcal{A} then clearly $(-)(-)'$ has order 2 and is a homomorphism on addition. Furthermore

$$\begin{aligned} (-)(-)' \omega_{m,j}(a_0, \dots, a_{u-1}, a_u, a_{u+1}, \dots, a_m) &= (-) \omega_{m,j}(a_0, \dots, a_{u-1}, (-)'a_u, a_{u+1}, \dots, a_m) \\ &= \omega_{m,j}(a_0, \dots, a_{u-1}, (-)(-)'a_u, a_{u+1}, \dots, a_m) \end{aligned} \quad (4.2)$$

Also $a_0 \preceq a_1$ implies $(-)'a_0 \preceq (-)'a_1$, so $(-)(-)'a_0 \preceq (-)(-)'a_1$.

□

This raises the question of what to do when we want to identify negation maps, since too many negation maps may cause confusion. The natural solution is to identify a given negation map with the identity. Following [35] in spirit, given a negation map $(-)$, we can define an equivalence on \mathcal{A} by putting $a_1 \equiv a_2$ if $a_1 = (\pm)a_2$, i.e., $a_1 = a_2$ or $a_1 = (-)a_2$. (This is like taking the absolute value.)

Proposition 4.10. \equiv is an equivalence on \mathcal{A} , and $(-)$ becomes the identity map on \mathcal{A}/\equiv .

Proof. Reflexivity and symmetry are immediate, and transitivity is also clear, since if $a_1 = (\pm)a_2$ and $a_2 = (\pm)a_3$, then $a_1 = (\pm)a_3$. Hence \equiv is an equivalence.

□

(But addition need not be well-defined on \mathcal{A}/\equiv . We will return to this issue in §6.9.)

Corollary 4.11. *Given two negation maps $(-)$ and $(-)'$, applying the equivalence of Proposition 4.10 to the negation map $(-)(-)'$, we may identify $(-)$ and $(-)'$ on the image of \mathcal{A} .*

Lemma 4.12. *If $(-)$ is a negation map on \mathcal{A} and $(-)'$ is a negation map on \mathcal{A}' , then $(-) \times (-)'$ is a negation map on $\mathcal{A} \times \mathcal{A}'$.*

Proof. By components.

□

5. SYSTEMS

Having the main concepts at our disposal, let us set up the basic framework, which will encompass most of the forthcoming applications, including those in §6.10. The main structure of interest is the **system**, given in Definition 5.1.

5.1. A general overview of systems.

- We start with a carrier \mathcal{A} in universal algebra, a semigroup endowed with a negation map $(-)$, and perhaps with extra operators with which $(-)$ must be compatible. In the usual tropical applications \mathcal{A} often is a semiring[†], but could be only a pre-semiring[†] or even have nonassociative multiplication, as we shall see.
- Next, we specify the designated set $\mathcal{T}(\mathcal{A})$ of “tangible elements,” denoted simply as \mathcal{T} when there is no ambiguity, requiring that $(-)a \in \mathcal{T}$ for all $a \in \mathcal{T}$. We enlarge our signature to include \mathcal{T} . When $0 \in \mathcal{A}$, we write $\mathcal{T}_0 := \mathcal{T} \cup \{0\}$. Since tropical structures and hypergroups both focus on tangible elements, we want the tangible elements to play a special role. For convenience, we always assume that \mathcal{T}_0 generates $(\mathcal{A}, +)$ as a semigroup. Consequently, each multilinear operator is given by its action on \mathcal{T} .

When necessary, we make the technical assumption that for any $a \in \mathcal{T}$ there is some $c \in \mathcal{A}$ such that $a = a + c^\circ$. This is obvious if $0 \in \mathcal{A}$, and in general we just take c “small enough.”

- We attach **\preceq -surpassing identities** to our signature. These are universal sentences of the form

$$f(x_1, \dots, x_m) \preceq g(x_1, \dots, x_m).$$

Many of these come up naturally in matrix theory.

- Along this vein, a surpassing relation \preceq is required to be compatible with the other operators in the additive signature, including the negation map, satisfying the extra property:

$$\omega(a_0, \dots, a_m) \preceq \omega(a'_0, \dots, a'_m), \quad (5.1)$$

for each operator ω whenever $a_i \preceq a'_i \in \mathcal{A}$.

- We are about to pinpoint the structures of interest in this theory.

Definition 5.1. A **triple** is a collection $(\mathcal{A}, \mathcal{T}, (-))$, where $\mathcal{T} \subset \mathcal{A}$, \mathcal{T}_0 generates $(\mathcal{A}, +)$ as a semigroup, and $(-)$ is a negation map. It can be formulated in universal algebra, where \mathcal{T} is taken as part of the signature.

A **system** $(\mathcal{A}, \mathcal{T}, (-), \preceq)$ is a triple $(\mathcal{A}, \mathcal{T}, (-))$ together with a surpassing relation \preceq .

The system $(\mathcal{A}, \mathcal{T}, (-), \preceq)$ is of **k -th kind** ($k = 1, 2$) is a triple $(\mathcal{A}, \mathcal{T}, (-))$ if $(-)$ is a negation map of k -th kind.

- There is a delicate issue here, when we start with a semiring[†] R with negation map, and its triple $(R, \mathcal{T}, (-))$. We have seen how $(-)$ extends naturally to any module M , but the correct definition of $\mathcal{T}(M)$ may not be clear. One common instance is for $M = R^{(I)}$, described in Example 1.52.

5.2. Uniquely negated triples.

Recall **uniquely negated triple** from Definition 1.50. This is the first structure for which we can say something of interest. The next result is the key to the relationship between \mathcal{A} and \mathcal{T} .

Proposition 5.2. Assume uniqueness of the quasi-negative.

- (i) If $c(-)c = a + b$ for $a, b \in \mathcal{T}$ and $c \in \mathcal{A}$, then $b = (-)a$.
- (ii) If $a + c^\circ = b$ for $a, b \in \mathcal{T}$, then $b = a$.
- (iii) \preceq_\circ (Definition 1.47) restricts to equality on \mathcal{T} .

Proof. (i) By definition $b = (-)a$.

(ii) $(a + c)^\circ = b(-)a$, so apply (i).

(iii) By (ii). □

Corollary 5.3.

- (i) If $e' \in \mathcal{T}$, then $e' = 1$.
- (ii) If $0 \in \mathcal{T}$ then $\mathcal{T} \cap \mathcal{A}^\circ = \{0\}$.

Proof. (i) If $e' \in \mathcal{T}$, then $e'(-)\mathbb{1} \in \mathcal{A}^\circ$, implying $e' = 1$.

(ii) Suppose $a^\circ \in \mathcal{T}$ for $a \in \mathcal{A}$. Then $a^\circ + 0 = a^\circ \in \mathcal{T}^\circ$ but also $a^\circ + a^\circ \in \mathcal{T}^\circ$, so $a^\circ = 0$. \square

If the triple $(R, \mathcal{T}, (-))$ is uniquely negated, then so is the triple $(R^{(I)}, \mathcal{T}(R^{(I)}), (-))$ of the free module.

Remark 5.4. Corollary 5.3 yields $\mathcal{T}_0 \cap \mathcal{A}^\circ = \{0\}$, so $\mathcal{T} \subseteq \mathcal{A} \setminus \mathcal{A}^\circ$. Thus, when \mathcal{T} is not specified in prior, we could simply define $\tilde{\mathcal{T}} = \mathcal{A} \setminus \mathcal{A}^\circ$, the **default triple**.

Unique negation in the default means that if $a, b \notin \mathcal{A}^\circ$ with $a + b \in \mathcal{A}^\circ$, then $b = (-)a$. Put another way, if $a, b \in \tilde{\mathcal{T}}$ with $b \neq (-)a$, then $a + b \in \tilde{\mathcal{T}}$, so the default triple is meta-tangible if it is uniquely negated.

Lemma 5.5. In the default triple, \mathcal{T} contains every multiplicative subgroup of \mathcal{A} .

Proof. The ideal \mathcal{A}° cannot contain any invertible element. \square

When $1 \in \mathcal{T}$, we incorporate the elements e, e' from Definition 1.20 into the signature. Many of the applications make \mathcal{T} a proper submonoid or subgroup, in which case we put the appropriate operators and identical relations into the signature. Later we forego this condition when discussing Grassmann algebras and nonassociative algebras.

5.2.1. \mathcal{T} -Reversible systems.

We are also interested in the converse of Lemma 1.48, which ties in with matroid theory.

Definition 5.6. A surpassing relation \preceq is called **\mathcal{T} -reversible** if $a \preceq b + c$ implies $b \preceq a(-)c$ for $a, b \in \mathcal{T}$.

A **\mathcal{T} -reversible system** is a system $(\mathcal{A}, \mathcal{T}, (-), \preceq)$ where the surpassing relation \preceq is \mathcal{T} -reversible.

5.2.2. \mathcal{T} -matroidal systems.

We now introduce a definition which, though weaker than meta-tangibility and thus not so relevant for tropical algebra, is quite relevant for wider classes of hypergroups.

Definition 5.7. A system is called **\mathcal{T} -matroidal** (or just **matroidal** if \mathcal{T} is understood) if it satisfies the following property, for $a \in \mathcal{T}$, and $c \in \mathcal{A}$:

$$a + c \in \mathcal{A}^\circ \text{ implies } (-)a \preceq c. \quad (5.2)$$

Proposition 5.8. Any matroidal system $(\mathcal{A}, \mathcal{T}, (-), \preceq)$ is uniquely negated. It is \mathcal{T} -reversible when $\preceq = \preceq_\circ$.

Proof. If $a(-)b \in \mathcal{A}^\circ$ then $(-)a \preceq (-)b$ by (5.2), implying $a \preceq b$, and symmetrically $b(-)a = (-)(a(-)b) \in \mathcal{A}^\circ$, implying $b \preceq a$. Hence quasi-negatives are unique.

For reversibility, suppose that $a \preceq_\circ b + c$, i.e., $a + d^\circ = b + c$. Then $a(-)a + d^\circ = b(-)a + c = b + (c(-)a)$, so $(-)b \preceq_\circ c(-)a$, i.e., $b \preceq_\circ a(-)c$. \square

Example 5.9. Given a uniquely negated system $(\mathcal{A}, \mathcal{T}_0, (-), \preceq)$, take $M = \mathcal{A}^{(m)}$, $\mathcal{T}(M) = \mathcal{T}_0^{(m)}$, with $(-)$ and \preceq defined componentwise. Then $(M, \mathcal{T}(M), (-), \preceq)$ also is a uniquely negated system of the same kind, although not meta-tangible, which plays an important role in Sections 8.5, 9, and 11.

If $(\mathcal{A}, \mathcal{T}, (-), \preceq)$ is a matroidal system, then $(\mathcal{A}^{(I)}, \mathcal{T}^{(I)}, (-), \preceq)$ is also a $\mathcal{T}^{(I)}$ -matroidal system.

5.3. The characteristic of a triple.

Here is another basic notion taken from classical algebra, although its role in tropical theory is indirect.

Definition 5.10. A semigroup $(M, +)$ with a distinguished constant $\mathbf{1}$ has **characteristic** k if $\mathbf{k} + \mathbf{1} = \mathbf{1}$, with $k \geq 1$ minimal.

M has **characteristic** 0 if M does not have characteristic k for any $k \geq 1$.

Example 5.11.

- (i) M has characteristic 1 iff it is idempotent.
- (ii) For $(-)$ of the first kind, $M = \mathcal{A}$ has characteristic 1 or 2 iff $e' \in \mathcal{T}$, since in this case $e' = \mathbf{3}$.
- (iii) Suppose $\mathbf{m} = \mathbf{m}'$ for some $m < m'$. For m minimal such, the \mathbf{j} are distinct for all $j \leq m$, and then the \mathbf{j} comprise a cycle with period $m' - m$. (When $m = 1$, this is precisely the definition of characteristic $m' - m$. But one could have characteristic 0 with $m > 1$, in which case we call this process **cycling**, as illustrated in Example 1.29.)

Lemma 5.12. If M has characteristic k and $\mathbf{m} + \mathbf{1} = \mathbf{1}$, then k divides m .

Proof. A standard Euclidean algorithm argument. Write $m = qk + r$, where $0 \leq r < k$. By definition $m \geq k$, and $\mathbf{r} + \mathbf{1} = \mathbf{qk} + \mathbf{r} + \mathbf{1} = \mathbf{m} + \mathbf{1} = \mathbf{1}$. But $r < k$, so we must have $r = 0$. \square

5.4. Neutral elements. From our point of view, e “almost” is a zero element. If we want to compare e with $\mathbf{0}$ (to see how far our triple is from being classical), there are two properties that need to be checked.

Definition 5.13. In a semigroup $(\mathcal{A}, +)$ with tangible elements \mathcal{T} and negation map $(-)$, an element $b \in \mathcal{T}$ is **a -neutral** if $b = b + a^\circ$.

\mathcal{A} is **a° -neutral** if every element in \mathcal{T} other than $(\pm)a$ is a° -neutral.

\mathcal{A} is **e -absorbing** if $e \preceq ae$ for all $a \in \mathcal{T}$. The **e -absorbing part** of \mathcal{T} is $\{a \in \mathcal{T} : e \preceq ae\}$.

The absorbing property often is automatic:

Lemma 5.14. When \mathcal{T} is a group, if \mathcal{A} is an a° -neutral \mathcal{T} -premodule, then, before adjoining $\mathbf{0}$, \mathcal{A} is already a \mathcal{T} -module with zero element a° .

Proof. Multiply through by a^{-1} , to show that \mathcal{A} is e -neutral. This means for every $c \neq (\pm)\mathbf{1}$ that $e + c = c$, and thus $e + c^\circ = c^\circ$. But then $e + c^{-1} = c^{-1}$, implying $c^\circ + \mathbf{1} = \mathbf{1}$, and thus $c^\circ = c^\circ + e = e$, for all c . But this means $ce = c^\circ = e$ for all $c \neq (\pm)e$, and clearly $(\pm)ee = e$, so e is absorbing, and furthermore $a(-)a = e$ for all $a \in \mathcal{A}$. We conclude that e is the zero element of \mathcal{A} . \square

Corollary 5.15. Suppose \mathcal{A} is uniquely negated, a° -neutral, and \mathcal{T} -invertible. Then \mathcal{A} is a ring with $a^\circ = \mathbf{0}$.

5.5. Polynomials and their roots.

One defines polynomials $\mathcal{A}[\lambda]$ in the usual way over a semiring \mathcal{A} , the product being the familiar convolution product. We write λ for $\mathbf{1}\lambda$. Given a system $(\mathcal{A}, \mathcal{T}, (-), \preceq)$, one should note that $\mathcal{T}[\lambda]$ is not closed under multiplication, since for example $(\lambda(-)\mathbf{1})^2 = \lambda^2 + e\lambda + \mathbf{1}$.

Hence, one identifies polynomials in terms of their values as functions; for example, over any meta-tangible system, $(\lambda(-)\mathbf{1})^2 = \lambda^2 + e\lambda + \mathbf{1}$. We say that two polynomials $f(\lambda_1, \dots, \lambda_n)$ and $g(\lambda_1, \dots, \lambda_n)$ are **\mathcal{T} -equivalent**, temporarily written $f \sim g$, if $f(\mathbf{a}) = g(\mathbf{a})$ for all $\mathbf{a} = (a_1, \dots, a_n) \in \mathcal{T}_0^{(n)}$. In this sense, $(\mathcal{A}[\lambda]/\sim, \mathcal{T}[\lambda]/\sim, (-), \preceq)$ becomes a uniquely negated system, where $(-)[\sum a_i \lambda^i] = [\sum ((-)a_i) \lambda^i]$. (One does the same for Laurent series, rational functions, etc.) These are important in affine geometry, and are uniquely negated and \mathcal{T} -reversible and cancellative when \mathcal{A} is, but fail to be meta-tangible since for example $(\lambda + \mathbf{1}) + (\lambda(-)\mathbf{1}) = \mathbf{2}\lambda + e$ is not tangible.

Generalizing [47, 51], we say that \mathbf{a} is a (systemic) **root** of f if $f(\mathbf{a}) \in A^\circ$. This gives rise to affine geometry over systems.

5.5.1. Localization.

There is a standard technique of commutative localization, which we can use for passing from cancellative meta-tangible systems to invertible meta-tangible systems. (We defer noncommutative localization for future work.) Localization by a submonoid S is a common tool in universal algebra; cf. [14] for multiplicative monoids. One defines the equivalence $(s_1, a_1) \equiv (s_2, a_2)$ when $s(s_1 a_2) = s(s_2 a_1)$ for some $s \in S$. In the cancellative case, we can dispose of s .

Here we localize \mathcal{A} with respect to multiplicative submonoids S of \mathcal{T} , extending the various operators via the rule

$$\omega_m(a_1, \dots, s_k^{-1} a_k, \dots, a_m) = s_k^{-d} \omega_m(a_1, \dots, a_k, \dots, a_m)$$

where ω_m is k -homogeneous of degree d . (In particular, $s_1^{-1} a_1 s_2^{-1} a_2 = (s_1 s_2)^{-1} a_1 a_2$ and

$$s_1^{-1} a_1 + s_2^{-1} a_2 = (s_1 s_2)^{-1} (s_2 a_1 + s_1 a_2),$$

or equivalently, seen via common denominators,

$$s^{-1} a_1 + s^{-1} a_2 = s^{-1} (a_1 + a_2).$$

The ensuing system is denoted $(S^{-1}\mathcal{A}, S^{-1}\mathcal{T}, (-), \preceq)$, where

$$(-)(s^{-1}a) := s^{-1}((-)a), \quad s \in S.$$

Lemma 5.16. (i) $s^{-1}((-)a) = ((-)s)^{-1}a$.
(ii) $(s^{-1}a)^\circ = s^{-1}a^\circ$.

Proof. (i) Cross multiply to get $sa = (-)(-)sa$.

(ii) $(s^{-1}a)^\circ = s^{-1}a(-)s^{-1}a = s^{-1}(a(-)a) = s^{-1}(a^\circ)$. \square

Proposition 5.17. *If $(\mathcal{A}, \mathcal{T}(\mathcal{A}), (-), \preceq)$ is a uniquely negated system, with S a multiplicative submonoid of \mathcal{T} , then the system $(S^{-1}\mathcal{A}, S^{-1}\mathcal{T}(\mathcal{A}), (-), \preceq)$ also is uniquely negated.*

Proof. We need to verify unique inverses. Suppose $s_1^{-1}a_1$ is a quasi-negative of $s^{-1}a$. Then

$$(ss_1)^{-1}(sa_1 + s_1a) = s_1^{-1}a_1 + s^{-1}a \in (S^{-1}\mathcal{A})^\circ,$$

implying $sa_1 + s_1a \in \mathcal{A}^\circ$, and thus $sa_1 = (-)s_1a = s_1((-)a)$, and $s_1^{-1}a_1 = s^{-1}((-)a)$, which is uniquely defined. \square

In particular, if \mathcal{T} is a cancellative monoid with $\mathcal{T}(\mathcal{A}) = \mathcal{T}$, then taking $S = \mathcal{T}$ we can localize to the group $\mathcal{T}^{-1}\mathcal{T}$.

5.6. Triples with involution.

Involutions are so important in classical algebra, that we look for their systemic analog.

Definition 5.18. *An **involution** on a triple $(\mathcal{A}, \mathcal{T}, (-))$ is an anti-isomorphism of degree 2, i.e., an additive homomorphism satisfying $(\forall \alpha \in F, a, a_i \in \mathcal{T})$:*

- $(a^*)^* = a$,
- $(a_1 a_2)^* = a_2^* a_1^*$,
- $(\alpha a)^* = \alpha(a^*)$,
- $((-)a)^* = (-)a^*$.

Lemma 5.19. $(c^\circ)^* = (c^*)^\circ$.

Proof. $(c^\circ)^* = (c(-)c)^* = c^*(-)c^* = (c^*)^\circ$. \square

Remark 5.20. *When dealing with systems, we also require that if $b_1 \preceq b_2$ in \mathcal{A} , then $b_1^* \preceq b_2^*$. This is automatic for \preceq_\circ , since $(b_1 + c^\circ)^* = b_1^* + (c^*)^\circ$.*

Example 5.21. *Examples of involutions on matrices over a semialgebra \mathcal{A} :*

- (i) *The transpose map on $M_n(\mathcal{A})$ is an involution denoted by $A \mapsto A^t$.*
- (ii) *When $n = 2m$ and \mathcal{A} has a negation map, there is another involution, called the **symplectic** involution (s) , given by $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^s = \begin{pmatrix} A_{22}^t & (-)A_{12}^t \\ (-)A_{21}^t & A_{11}^t \end{pmatrix}$, where the $A_{ij} \in M_m(\mathcal{A})$.*

Semialgebras with involution come up in many of our subsequent examples. The involution can be expressed as a unary operator in universal algebra, in which case we also require

$$\omega(a_1, \dots, a_m)^* = \omega(a_1^*, \dots, a_m^*)$$

for operators ω other than multiplication, and $(*)$ is notated together with R , as $(R, *)$.

Lemma 5.22. *Define $(R, *)^+ := \{r^* + r : r \in R\}$ and $(R, *)^- := \{r^*(-)r : r \in R\}$. These sets respectively are symmetric and antisymmetric.*

Proof. $(r^* + r)^* = r + r^* = r^* + r$, and $(r^*(-)r)^* = r + ((-)r)^* = r(-)r^* = (-)(r^*(-)r)$. \square

5.7. Hyperstructures.

Let $(G, \boxplus, 0)$ be a hypergroup. In all of Baker's examples [7, Examples 2.8–2.12], the function $a \mapsto -a$ in [7] is a negation map with unique quasi-negatives, enabling one to view hyperstructures within the framework of this paper. (In [7, Examples 2.8, 2.9, 2.12] the negation map actually is the identity, whereas in [7, Examples 2.10, 2.11] it is the usual negative.) In fact, we have the \mathcal{T} -semiring $(\mathcal{P}(G), \mathcal{T})$, where \mathcal{T} is the set of singletons of $\mathcal{P}(G)$, which can be identified with G .

Definition 5.23. *Given a hypergroup \mathcal{T} , we define $\tilde{\mathcal{T}}$ to be the sub-semigroup of $\mathcal{P}(\mathcal{T})$ generated by the singletons (which we identify with \mathcal{T}). The **system** of a hypergroup \mathcal{T} is $(\tilde{\mathcal{T}}, \mathcal{T}, (-), \subseteq)$ where we start with the power semigroup $(\mathcal{P}(\mathcal{T}), \boxplus)$ and take $(-)$ is as in Lemma 1.40. \preceq now is given by $S_1 \preceq S_2$ iff $S_1 \subseteq S_2$.*

This gives us our other major example of a surpassing relation.

Proposition 5.24. *The system $(\tilde{\mathcal{T}}, \mathcal{T}, (-), \subseteq)$ of a hypergroup \mathcal{T} is both uniquely negated and \mathcal{T} -reversible.*

Proof. The relation \subseteq clearly satisfies $S \subseteq \{a\}$ iff $S = \{a\}$, and likewise it is a PO, so \subseteq is a surpassing relation. The last assertion is by definition of negation and reversibility in the hypergroup \mathcal{T} . \square

The hypergroup morphisms turn out to be \preceq -morphisms in the sense of §8.2 below, which then yields theorems about hypergroups.

Example 5.25. *Let R be a commutative semiring with a negation map $(-)$. Any multiplicative monoid A , together with a surjection of multiplicative monoids $\varphi : R \rightarrow A$, has an induced hyperring structure given by the hyperaddition law*

$$a_1 \boxplus a_2 := \varphi(\varphi^{-1}(a_1) + \varphi^{-1}(a_2)).$$

This extends naturally to $\mathcal{P}(A)$, via

$$S_1 \boxplus S_2 := \varphi(\varphi^{-1}(S_1) + \varphi^{-1}(S_2)).$$

Distributivity on $\mathcal{P}(A)$ is inherited from distributivity on R . Explicitly, if $(\boxplus_i a_i)(\boxplus_j b_j) \in (\boxplus S)(\boxplus T)$, then this is

$$\sum_i \varphi(\varphi^{-1}(a_i)) \sum_j \varphi(\varphi^{-1}(b_j)) = \sum_{i,j} \varphi(\varphi^{-1}(a_i) \varphi^{-1}(b_j)) = \sum_{i,j} \varphi(\varphi^{-1}(a_i b_j)) \in \boxplus(ST). \quad (5.3)$$

Given $\boxplus_{i,j} a_i b_j \in \boxplus(ST)$, we reverse (5.3) to get

$$\sum_{i,j} \varphi(\varphi^{-1}(a_i b_j)) = \sum_i \varphi(\varphi^{-1}(a_i)) \sum_j \varphi(\varphi^{-1}(b_j)) \in (\boxplus S)(\boxplus T).$$

Thus $\mathcal{P}(A)$ is a semiring, and its theory can be embedded into semiring theory.

Example 5.26. *As a special case of Example 5.25, let R be a commutative semiring with a negation map $(-)$, and with a given multiplicative subgroup U . The surjection of multiplicative monoids $\varphi : R \rightarrow R/U$ has an induced additive structure given by the hyperaddition law*

$$[a_1] \boxplus [a_2] := \{u_1 a_1 + u_2 a_2 : u_i \in U\}.$$

This extends naturally to $\mathcal{P}(A)$, via

$$S_1 \boxplus S_2 := \{u_1 a_1 + u_2 a_2 : u_i \in U, a_i \in S_i\}.$$

Distributivity on $\mathcal{P}(A)$ and thus on \hat{A} is inherited from distributivity on R . Thus $\mathcal{P}(A)$ is a semiring, and its theory can be embedded into semiring theory.

Furthermore $[a] = 0$ iff $a = 0$, so $0 \in [a_1] + [a_2]$ iff $Ua_1 \cap -Ua_2 \neq \emptyset$, iff $Ua_1 = -Ua_2$. (Here we rely on U being a group.) But this is true iff $[a_1] = -[a_2]$, so R/U is a hyperring under this addition.

A mild surprise: The system $(\widetilde{R/U}, R/U, (-), \subseteq)$ is meta-tangible iff $a_1 - Ua_2 = U(a_1 - a_2)$ for all $a_1 \neq a_2$, which is true only in special situations such as the sign hyperfield and Krasner's hyperfield described in Examples 12.8. So in a sense this example is "too" classical.

Many of the systems ensuing from Example 5.25 are meta-tangible, as we shall see in Examples 12.8 below.

Definition 5.27. A hypergroup \mathcal{T} is $(-)$ -closed if $a + b \in \mathcal{T}$ whenever $a \neq -b$; \mathcal{T} is $(-)$ -bipotent if $a + b \in \{a, b\}$ whenever $a \neq -b$.

Lemma 5.28. The hypergroup \mathcal{T} is meta-tangible, resp. closed, iff the system $(\widetilde{\mathcal{T}}, \mathcal{T}, (-), \subseteq)$ is meta-tangible, resp. $(-)$ -bipotent.

Proof. The definitions match. \square

Thus, in some ways the theory of hypergroups and hyperrings embeds into the theory of systems over \mathcal{T} -semirings[†].

We would like to make this more formal when discussing categories of systems, but certain subtleties arise in these considerations. In order not to throw the paper off balance, we defer more details about hypergroups and hyperfields as systems, and their categories until Theorem 8.9.

6. META-TANGIBLE TRIPLES AND THEIR SYSTEMS

This section deals in depth with meta-tangible triples $(\mathcal{A}, \mathcal{T}, (-))$, which are uniquely negated triples having a stronger tropical flavor. We work with $0 \in \mathcal{T}_0$, and assume that our triple is regular (§2.7), in order to avoid technical issues in universal algebra. Eventually we show in Theorem 6.29 that \leq_\circ rounds out the system (although there are a few other possible surpassing relations).

The key property is:

Lemma 6.1. Any meta-tangible triple $(\mathcal{A}, \mathcal{T}, (-))$ satisfies $a + b \in \mathcal{T}$ whenever $a, b \in \mathcal{T}$ with $b \neq (-)a$.

Proof. If $a + b \notin \mathcal{T}$, then $a + b \in \mathcal{A}^\circ$, so $a = (-)b$ by uniqueness of quasi-negatives. \square

Decisive results are available with tropical applications, but which do not hold for hyperfields in general.

Remark 6.2. We want to compute in a meta-tangible triple $(\mathcal{A}, \mathcal{T}, (-))$ via \mathcal{T} . Since $\mathcal{T} + \mathcal{T} \subseteq \mathcal{T}^+$, any meta-triple has the sub-triple $\mathcal{T} + \mathcal{A}^\circ$, so we may replace \mathcal{A} by $\mathcal{T} + \mathcal{A}^\circ$.

Almost conversely, if $\mathcal{T} \cup \mathcal{A}^\circ = \mathcal{A}$, then for any $a \neq (-)b$ we must have $a + b = c$ for $c \in \mathcal{T}$, and thus the triple $(\mathcal{A}, \mathcal{T}, (-))$ is meta-tangible.

On the other hand, if $a^\circ, b^\circ \in \mathcal{T}^\circ$, then $a^\circ + b^\circ = (a + b)^\circ$, which is in $(\mathcal{T}^+)^\circ = \mathcal{T}^\circ \cup (\mathcal{T}^\circ)^\circ$. Continuing inductively on height, we have

$$\mathcal{A}^\circ = \mathcal{T} \cup \mathcal{T}^\circ \cup (\mathcal{T}^\circ)^\circ \cup \dots$$

What about $a + b^\circ$? If $a \neq (\pm)b$ this is $(a + b)(-)b$ which is either tangible or $b(-)b = b^\circ$. Thus, we need to consider elements $(\pm)a + a^\circ$, which is $(\pm)ae'$ when $1 \in \mathcal{T}$. In this case, we can understand the structure of \mathcal{A}° after we know the element e' .

Proposition 6.3. One of the following must hold, in a meta-tangible triple \mathcal{A} containing 1 :

- (i) $e' = 1$.
- (ii) $e' = e$.
- (iii) $(-)1 = 1$, so $e' = 3$.

Proof. If e' is tangible then $e' = 1$ by Proposition 5.2(ii). Thus we may assume that $e' = 2(-)1$ is not tangible. If 2 is tangible then $2 = 1$, implying $e' = e$. If 2 is not tangible then $(-)1 = 1$. \square

Lemma 6.4. Suppose $a_i \in \mathcal{T}$ with $\sum_{i=1}^{k-1} a_i \in \mathcal{T}$ but $\sum_{i=1}^k a_i \notin \mathcal{T}$. Then $\sum_{i=1}^{k-1} a_i = (-)a_k$.

Proof. $\sum_{i=1}^k a_i \in \mathcal{T}$ unless $\sum_{i=1}^{k-1} a_i = (-)a_k$, by Lemma 6.1. \square

Proposition 6.5. *Suppose $\sum_{i=1}^t a_i \in \mathcal{A}^\circ$. Then for some $k < t$, $a_k = (-)\sum_{i=1}^{k-1} a_i$.*

Proof. Take $k < t$ minimal satisfying Lemma 6.4. \square

Often we start with a semifield[†] F ; \mathcal{A} is an F -module, and \mathcal{T}_0 is “almost” a cone in the sense that if $a_1, a_2 \in \mathcal{T}_0$ with $\alpha_2 a_2 \neq (-)\alpha_1 a_1$ and $\alpha_1, \alpha_2 \in F$, then $\alpha_1 a_1 + \alpha_2 a_2 \in \mathcal{T}_0$.

Here is a surprisingly strong observation.

Lemma 6.6. *One of the following must hold, for $a, b \in \mathcal{T}$ in a meta-tangible triple:*

- (i) $(-)$ is of the first kind, and $a = b$.
- (ii) $a + b = a$ (and thus $a^\circ + b = a^\circ$).
- (iii) $a^\circ + b = b$.

Proof. First assume that $a = (-)b$. Also assume for the moment that $a(-)b = a + a \in \mathcal{T}$. We are done if $a(-)b = a$. But otherwise $(a(-)a)(-)b \in \mathcal{T}$, and thus equals $(-)b$, yielding (iii). We may assume that $a(-)b \notin \mathcal{T}$, implying $(-)a = b = a$, i.e., $(-)$ is of the first kind.

So we may assume that $a \neq (-)b$, implying $a + b \in \mathcal{T}$. If $a + b \neq a$, then $a^\circ + b = (a + b)(-)a \in \mathcal{T}$. Hence $b = a^\circ + b$ by Proposition 5.2(ii). \square

We saw in Corollary 5.3(i) that $\mathcal{T} \cap \mathcal{T}^\circ = \{0\}$ when $0 \in \mathcal{T}$. Let us strengthen this.

Corollary 6.7. *If $\mathcal{T} \cap \mathcal{A}^\circ \neq \{0\}$, then \mathcal{T} indeed contains the zero element 0.*

Proof. Suppose $b \in \mathcal{T} \cap \mathcal{A}^\circ$. Then $b = (-)b$. Moreover, for any $a \in \mathcal{T}$ we have $(a + b)(-)a = a^\circ + b = (a + b)^\circ \in \mathcal{A}^\circ$, so either $a = (-)b = b$ or $a + b = a$. This means $a + b = a$ for all $a \neq b \in \mathcal{T}$.

We claim that $a + b = a$ for all $a \in \mathcal{A}$. and $a = \sum a_i$, for $a_i \in \mathcal{T}$. If $a_i + b = a_i$ for some i , then clearly $a + b = a$. Hence we may assume that each $a_i = b$.

Thus it remains to show that $b + b = b$. and thus $a_i + d = b$. Write $b = d + d$ with $d = \sum_{j=1}^t d_j$ and each $d_j \in \mathcal{T}$, t minimal. If $t = 1$, i.e., $d \in \mathcal{T}$ then either $b = d$ or $b + d = d$ or $b + b = b + d^\circ = b$, and in each case we get $b + b = b$. Thus we are done unless $t \geq 2$. But if $d_i \neq d_j$ then we can replace $d_i + d_j$ by another element of \mathcal{T} and lower t , and conclude by induction. Thus we may assume that $d_1 = d_2$ and again have either $b = d_1$ or $b + d_1 = d_1$ or $b + d_1^\circ = b$, and in each case we get $b + b = b$.

We conclude that b is the zero element. (Note that if $0 \in \mathcal{T}$ then $0 = 0^\circ \in \mathcal{T} \cap \mathcal{T}^\circ$ by Lemma 1.16.) \square

Remark 6.8. *We cannot have both (ii) and (iii) in Lemma 6.6 unless $a^\circ = b = 0$.*

If $a + b = a$ then $a^\circ + b = a(-)a = a^\circ$, and if also $a^\circ + b = b$ then $a^\circ = a^\circ + b = b \in \mathcal{T} \cap \mathcal{T}^\circ = \{0\}$.

Lemma 6.9. *Any triple has the partial pre-order $<^\circ$ given by $a_1 <^\circ a_2$, iff $a_1^\circ = a_2^\circ$ or $a_1^\circ + a_2^\circ = a_2^\circ$. Then $(-)a_1 \leq^\circ a_1$. Hence, $a_1 <^\circ a_2$ iff $a_1 <^\circ (-)a_2$, iff $(-)a_1 <^\circ a_2$.*

Proof. Suppose $a_1^\circ + a_2^\circ = a_2^\circ$ and $a_2^\circ + a_3^\circ = a_3^\circ$. Then

$$a_1^\circ + a_3^\circ = a_1^\circ + (a_2^\circ + a_3^\circ) = (a_1^\circ + a_2^\circ) + a_3^\circ = a_2^\circ + a_3^\circ = a_3^\circ.$$

The other verifications are similar. \square

Proposition 6.10. *For any meta-tangible triple, with $a_i \in \mathcal{T}$,*

- (i) $a_1^\circ + a_2^\circ \in \{a_1^\circ, a_2^\circ\}$, unless $a_1 = a_2$ and $(-)$ is of the first kind.
- (ii) In particular, $<^\circ$ restricts to a PO on \mathcal{A}° .
- (iii) (The trio property) When $(-)$ is of the first kind, if $a_1 \neq a_2$ for $a_i \in \mathcal{T}$ and $a_3 := a_1 + a_2 \notin \{a_1, a_2\}$, then $a_{i+2} = a_{i+1} + a_i$ for each i , subscripts taken modulo i .

Proof. (i) by Lemma 6.6.

(ii) Follows at once from (i).

(iii) When $(-)a = a$, each of the last equations is equivalent, by unique negation, to $a_1 + a_2 + a_3 = a_3^\circ \in \mathcal{A}^\circ$. \square

An example of this phenomenon was given in Example 1.29(v). Note that the first assertion only required that \mathcal{A}° be ordered. There is a more technical version of Lemma 6.6 for b nontangible.

Proposition 6.11. *One of the following must hold, for $a \in \mathcal{T}$, any $b = \sum_{i=1}^t b_i$ for $b_i \in \mathcal{T}$ in a meta-tangible triple:*

- (i) $(-)$ is of the first kind, and $a = b \in \mathcal{T}$.
- (ii) $(-)$ is of the first kind, there is some $m \leq t$ and some j such that $b = \mathbf{m}b_j$ and $a = b_j$.
- (iii) $a + b = a$ (and thus $a^\circ + b = a^\circ$).
- (iv) $a^\circ + b = b$.

Proof. We may assume that $t > 1$, by Lemma 6.6. Also, applying Lemma 6.6 to each j , if we have some $b_i \neq (-)b_j$ we can take the sum $b_i + b_j \in \mathcal{T}$ and conclude by induction. Thus, for $t \geq 3$, the conclusion is clear unless $(-)$ is of the first kind and all b_i are equal, in which case we are done by Lemma 6.6.

So we may assume that $t = 2$. Clearly $b_1 \neq (-)b_2$ since a is tangible, so $b = b_1 + b_2 \in \mathcal{T}$, and again we are done by Lemma 6.6. \square

Here are two basic properties of meta-tangible triples.

Lemma 6.12. *For $a_1 \neq a_2$ in \mathcal{T} , $a_1 + a_2 = a_2$ implies $a_1(-)a_2 = (-)a_2$ (or equivalently, $a_2(-)a_1 = a_2$).*

Proof. We are done unless $a_2 \neq \mathbf{0}$. Then $a_1(-)a_2 \neq a_1$, since otherwise

$$a_2 = a_1 + a_2 = (a_1(-)a_2) + a_2 = (a_1 + a_2)(-)a_2 = a_2(-)a_2 \in \mathcal{T} \cap \mathcal{T}^\circ = \{\mathbf{0}\},$$

a contradiction. Hence $(a_1(-)a_2)(-)a_1 \in \mathcal{T}$, but $(a_1(-)a_2)(-)a_1 + a_2 = (a_1 + a_2)(-)(a_1 + a_2) = a_2(-)a_2$, so $(a_1(-)a_2)(-)a_1 = (-)a_2$ and $a_1(-)a_2 = a_1(-)(a_1 + a_2) = (a_1(-)a_2)(-)a_1 = (-)a_2$. \square

Lemma 6.13. *If $c := a + b \notin \{a, b\}$, then $a^\circ = b^\circ$ and $a + a(-)a = a$ and $b + b(-)b = b$.*

Proof. By Lemma 6.6, $c(-)a = b$, and likewise $c(-)b = a$. Hence

$$b^\circ = c(-)a(-)b = a^\circ,$$

implying $a = a + b(-)b = a + a(-)a$. \square

6.1. The characteristic of a meta-tangible triple.

In this subsection we assume that \mathcal{A} contains $\mathbf{1}$.

Lemma 6.14. *If $(-)$ is of the second kind and $e' \notin \mathcal{T}$, then \mathcal{A} is idempotent.*

Proof. $e' = \mathbf{2}(-)\mathbf{1}$ and by hypothesis $\mathbf{2} \in \mathcal{T}$, so $\mathbf{2} = \mathbf{1}$. Hence $a + a = \mathbf{2}a = \mathbf{1}a = a$ for all $a \in \mathcal{T}$. \square

Lemma 6.15. *Define $\mathbf{Z} = \{\pm a : a \in \mathbf{N}\}$. Then \mathbf{Z} is a sub-triple of \mathcal{A} , with $\mathcal{T}(\mathbf{Z}) = \mathcal{T} \cap \mathbf{Z}$.*

Proof. Suppose \mathcal{A} has characteristic k . $\mathbf{k} - \mathbf{1} = (-)\mathbf{1}$, by (since otherwise $\mathbf{k} \in \mathcal{T}$). Hence $\mathbf{k} = e$. Also $\mathbf{k} + \mathbf{1} = e'$, and thus $\mathbf{1}$ if $e' \in \mathcal{T}$. If $(-)$ is of the second kind and $e' = \mathbf{2}(-)\mathbf{1} \notin \mathcal{T}$, then $\mathbf{2} = \mathbf{1}$, so \mathcal{A} is idempotent. \square

Proposition 6.16. *Suppose that $(\mathcal{A}, \mathcal{T}, -)$ is a meta-tangible triple of the second kind. Either $\mathbf{Z} \subseteq \mathcal{T}$ or $(\mathcal{A}, \mathcal{T}, -, \preceq)$ has characteristic k for some $k \geq 1$.*

Proof. Assume that $\mathbf{n} \neq \mathbf{1}$ for each n . By induction, each $\mathbf{n} + \mathbf{1} \in \mathcal{T}$. \square

Lemma 6.17. *Suppose in a meta-tangible triple $(\mathcal{A}, \mathcal{T}, (-))$ that $\mathbf{k} - \mathbf{1} \in \mathcal{T}$ but $\mathbf{k} \notin \mathcal{T}$. Then $\mathbf{k} - \mathbf{1} = (-)\mathbf{1}$ and $\mathbf{k} = e$. Furthermore, if $(-)$ is of the second kind, then either \mathcal{A} is idempotent or $\mathbf{k} + \mathbf{1} = \mathbf{1}$, implying the characteristic of \mathcal{A} divides k .*

Proof. $\mathbf{k} - \mathbf{1} = (-)\mathbf{1}$, by Lemma 6.4. Hence $\mathbf{k} = e$. Also $\mathbf{k} + \mathbf{1} = e'$, and thus $\mathbf{1}$ if $e' \in \mathcal{T}$. If $(-)$ is of the second kind and $e' = \mathbf{2}(-)\mathbf{1} \notin \mathcal{T}$ then $\mathbf{2} = \mathbf{1}$, so \mathcal{A} is idempotent. \square

6.2. Cancellative meta-tangible triples.

One main object under consideration in this paper is a \mathcal{T} -invertible meta-tangible triple. Accordingly, throughout this subsection we assume throughout that $(\mathcal{A}, \mathcal{T}, -, \preceq)$ is a cancellative meta-tangible triple, in particular containing the element e' , and we have a major structure theorem. (Localizing via Proposition 5.17 would yield a \mathcal{T} -invertible meta-tangible triple.)

Theorem 6.18. *Any cancellative meta-tangible triple $(\mathcal{A}, \mathcal{T}, -)$ satisfies one of the following cases:*

- (i) \mathcal{A} is $(-)$ -bipotent.
- (ii) $e' = \mathbb{1}$, with one of the following two possibilities.
 - (a) $(-)$ is of the first kind, of characteristic 2. (In other words $e' = \mathbf{3} = \mathbf{1} = \mathbb{1}$.)
 - (b) $(-)$ is of the second kind, either of finite characteristic or with $\{\mathbf{m} : m \in \mathbb{N}\}$ all distinct.

Proof. If \mathcal{A} is not $(-)$ -bipotent, we have $a, b \in \mathcal{T}$ with $b \neq (-)a$ and $c = a + b \notin \{a, b\}$. By Lemma 6.13, $a = a + a(-)a$; canceling a yields $e' = \mathbb{1}$.

If $(-)$ is of the first kind, then $\mathbf{1} = e' = \mathbf{3}$, so \mathcal{A} has characteristic 2.

If $(-)$ is of the second kind, then either \mathcal{A} has characteristic 0 with all $\mathbf{m} \in \mathcal{T}$, or \mathcal{A} has characteristic 0 by Proposition 6.16.

If $\mathbf{m}' = \mathbf{m}$ for some $m' > m \in \mathbb{N}$, then $\mathbf{m}' - \mathbf{1} = \mathbf{m}'(-)\mathbb{1} = \mathbf{m} - \mathbb{1} = \mathbf{m} - \mathbf{1}$, so iterating $m' - 1$ times yields $\mathbf{1} = \mathbf{m}'(-)(\mathbf{m} - \mathbf{1}) = \mathbf{k}$ for $k = m' - m$. \square

There are examples for each of these conclusions. The classical triple satisfies $e' = \mathbb{1}$, but is not $(-)$ -bipotent. The ELT triple (Example 6.58(ii)) satisfies both conditions and is of the second kind. The triple of the standard supertropical algebra is $(-)$ -bipotent of first kind, but only satisfies $e' = e$. Example 1.29(i) is $(-)$ -bipotent of the first kind, even failing $e' \in \mathcal{T}^+$.

Another instance of $e' = \mathbb{1}$ is given in Example 1.29 (v).

Example 6.19. *In Example 1.29(v), take $L = \{0, 1, \ell, \ell + 1\}$ to be the finite field of 4 elements. Although not $(-)$ -bipotent (since $\mathbb{1} + \ell = (\ell + \mathbb{1})$), the layered algebra $\mathcal{A} = L \times \mathcal{G}$ is meta-tangible of first kind, characteristic 2, and layer 2, satisfying $e' = \mathbf{3} = \mathbb{1}$. This comes up naturally in the classification (Case (1b) of Theorem 6.57).*

Corollary 6.20. *Suppose $(\mathcal{A}, \mathcal{T}, -, \preceq)$ is a cancellative meta-tangible triple of the second kind. Then \mathcal{A} is $(-)$ -bipotent iff $\mathbb{1} + \mathbb{1} = \mathbb{1}$.*

Proof. (\Rightarrow) Since $(-)\mathbb{1} \neq \mathbb{1}$, we have $\mathbb{1} + \mathbb{1} \in \{\mathbb{1}, \mathbb{1}\} = \{\mathbb{1}\}$.

(\Leftarrow) $e' = \mathbb{1}(-)\mathbb{1} = e$, so $e' \neq \mathbb{1}$, implying \mathcal{A} is $(-)$ -bipotent by Theorem 6.18. \square

Here is a cute application of Theorem 6.18, inspired by [29], who call such a situation in fuzzy rings “field-like.”

Corollary 6.21. *Suppose a, b are both tangible in an invertible meta-tangible triple $(\mathcal{A}, \mathcal{T}, (-))$. Then either there is $c \in \mathcal{T}$ such that $a + b + c \in \mathcal{A}^\circ$, or \mathcal{A} is a field.*

Proof. This is true by definition (taking $c = (-)(a + b)$) unless $b = (-)a$. Then $a + b = a^\circ$, so we are done if there is some c such that $a^\circ + c \in \mathcal{A}^\circ$; thus we may assume whenever $c \neq a$ that $a + c \neq a$, so $a^\circ + c = c$.

By Corollary 5.15, \mathcal{A} is a ring, and \mathcal{T}_0 is clearly closed under addition, implying $\mathcal{A} = \mathcal{T}_0$ is a field. \square

What can one say about the converse? Presumably not much, because of the wealth of examples of meta-tangible triples. But one could limit these if one strengthens the property of unique negation to:

If $a + b \in \mathcal{A}^\circ$ for $a \in \mathcal{T}$, then $b = (-)a$.

But this is too restrictive to be useful. Indeed, $(-)a + (a + b^\circ) \in \mathcal{A}^\circ$ for all $a \in \mathcal{T}$, implying $a + b^\circ = a$, so we conclude by Lemma 5.14 (when \mathcal{T} is a group) that \mathcal{A} is a ring.

6.2.1. Meta-tangible triples of positive characteristic.

Corollary 6.22. *Suppose $(\mathcal{A}, \mathcal{T}, (-))$ is a meta-tangible triple, such that \mathcal{A} has characteristic $k > 0$. Then one of the following possibilities holds:*

- (i) $k = 1$, i.e., \mathcal{A} is idempotent.

- (ii) $k > 1$, with $\mathbf{k} - \mathbf{1} = (-)\mathbf{1}$, and $\mathbf{k} = e$.
- (iii) $(-)$ is of the first kind and k is even. Furthermore, if $k > 2$, then \mathcal{T} is $(-)$ -bipotent.

Proof. By definition, $\mathbf{k} + \mathbf{1} = \mathbf{1}$, and we take $k + 1$ minimal such.

First assume that $(-)$ is of the second kind. If \mathcal{T} is also $(-)$ -bipotent then $\mathbf{1} + \mathbf{1} = \mathbf{1}$, so we have (i). Otherwise $\mathbf{2}(-)\mathbf{1} = e' = \mathbf{1}$ by Theorem 6.18, implying $\mathbf{j}(-)\mathbf{1} = \mathbf{j} - \mathbf{1}$ for all $j > 1$. Take k' minimal such that $\mathbf{k}' \notin \mathcal{T}$. (Clearly $k' > 1$, and $k' \leq k$.) Then by Lemma 6.17, $\mathbf{k}' - \mathbf{1} = (-)\mathbf{1}$ and $\mathbf{k}' = e$. By meta-tangibility, $\mathbf{k}' = (-)\mathbf{1}$, and $\mathbf{k}' + \mathbf{1} = \mathbf{1}(-)\mathbf{1} = e$, and we have (ii). This concludes the proof when $(-)$ is of the second kind.

Thus we may assume that $(-)$ is of the first kind. If k is odd then $\mathbf{1} = \mathbf{k} + \mathbf{1} = \frac{k+1}{2}e \in \mathcal{A}^\circ$, a contradiction, so k is even. The last assertion is a restatement of Theorem 6.18. \square

We also tie a loose thread from Proposition 4.10.

Lemma 6.23. *If \equiv is the equivalence of Proposition 4.10, and \mathcal{T} is $(-)$ -bipotent, then the corresponding triple of \mathcal{A}/\equiv is also $(-)$ -bipotent, under the induced addition and multiplication.*

Proof. We define

$$[a_1] + [a_2] := \begin{cases} [a_1 + a_2] & \text{if } a_1 \neq (\pm)a_2; \\ [a_1]^\circ & \text{if } a_1 = (\pm)a_2. \end{cases}$$

This is well-defined in view of Lemma 6.12, noting that $(-)a^\circ = a^\circ$. To check associativity and distributivity, it is enough to note for $a_i \in \mathcal{T}$ that if $a_1 + a_2 = a_1$ then

$$a_1^\circ + a_2 = (a_1(-)a_1) + a_2 = a_1 + a_2 = a_1 = a_1 + (a_1 + a_2);$$

if $a_1 + a_2 = a_2$ then

$$a_1^\circ + a_2 = (a_1(-)a_1) + a_2 = a_1 + a_2 = a_2 = a_1 + (a_1 + a_2).$$

\square

6.3. Uniform elements and height in meta-tangible triples.

Recall the definition of height from §1.2.1. We shall see that it often is easier to work with second kind than first kind, since then $a \neq (-)a$ implies $a + a \in \mathcal{T}$. We call a meta-tangible triple **exceptional** if it is $(-)$ -bipotent of first kind, of height > 2 . The main example is the layered triple (Example 1.28). Exceptional triples are sometimes a source of counterexamples, cf. Example 6.34.

Definition 6.24. *An element $c \in \mathcal{A}$ of height t is **uniform** if one of the following three possibilities occur, whenever we write $c = \sum_{i=1}^t a_i$:*

- (i) $t = 1$, i.e., $c \in \mathcal{T}$,
- (ii) c has height 2, i.e., $c = a^\circ$ for some $a \in \mathcal{T}$.
- (iii) The triple $(\mathcal{A}, \mathcal{T}, (-))$ is exceptional, and each $a_i = a_1$, in other words $c = \mathbf{t}a_1$, and $\mathbf{3} \neq \mathbf{1}$.

Theorem 6.25. *Every element of a \mathcal{T} -cancellative meta-tangible triple $(\mathcal{A}, \mathcal{T}, -)$ is uniform.*

Furthermore, if $c = \sum_{i=1}^t a_i \in \mathcal{T}$, then there is some $I \subseteq \{1, \dots, t\}$ such that $c = \sum_{i \in I} a_i$, such that $\sum_{i \in I \setminus \{j\}} a_i = c(-)a_j$ for each $j \in I$.

Proof. Induction on its height t' . Write $c = \sum_{i=1}^t a_i$. Take I minimal such that $c = \sum_{i \in I} a_i$. Then every subsum is in \mathcal{T} since if say $c' := \sum_{i \in I'} a_i = b^\circ \in \mathcal{T}^\circ$ then for $i \in I \setminus I'$ we cannot have $a_i + b = b$ (since then we could delete a_i), so $a_i + b = a_i$ and we could delete b .

Now for each j we have $c = a_j + \sum_{i \in I \setminus \{j\}} a_i$, implying

$$(a_j(-)c) + \sum_{i \in I \setminus \{j\}} a_i = c(-)c \in \mathcal{T}^\circ,$$

and thus $\sum_{i \in I \setminus \{j\}} a_i = (-)(a_j(-)c) = c(-)a_j$.

Thus we may assume that $t' \geq 2$. If $t' = 2$ and $a_1 \neq (-)a_2$ then $c \in \mathcal{T}$, so $a_1 = (-)a_2$ and $c = a_1^\circ$.

For $t' \geq 3$, if some $a_i \neq a_j$ then $a_i + a_j \in \mathcal{T}$ and we conclude by induction. Hence all $a_i = a_1$. If $\mathbf{3} = \mathbf{1}$ then we replace $a_1 + a_1 + a_1$ by a_1 and conclude by induction on height. Hence $(\mathcal{A}, \mathcal{T}, (-))$ is bipotent by Theorem 6.18, so it is exceptional. \square

Here is a general result about height, which could be used in coordination with Theorem 6.25.

Theorem 6.26. *Suppose that $\mathbf{t}a$ has height $u < t$ in a cancellative meta-tangible triple. Then one of the following holds:*

- (i) $\mathbf{t} = \mathbf{u}$.
- (ii) \mathcal{A} has some positive characteristic $k \leq t$.
- (iii) $2a \in \mathcal{T}$.

Proof. Write

$$\mathbf{t}a = \mathbf{u}b \tag{6.1}$$

for $b \in \mathcal{T}$ and $u < t$. Adding b to both sides gives $b + \mathbf{t}a = (\mathbf{u} + \mathbf{1})b$. If $a + b = b$ this reduces to $b = (\mathbf{u} + \mathbf{1})b$, so $\mathbf{u} + \mathbf{1} = \mathbf{1}$.

If $a + b = a$, adding a to both sides reduces (6.1) to $(\mathbf{t} + \mathbf{1})a = a$, so $k \leq t$. Now reduce the right side of (6.1) to get $\mathbf{t}a = \mathbf{u}'b$ for $u' \leq t$, and then on adding b on both sides until reach $u' = t$, which we reduce to $u' = \mathbf{1}$, i.e., $\mathbf{t}a = b$. Hence $a = (\mathbf{m} - \mathbf{1})b + \mathbf{t}a = \mathbf{m}b$ for all b . Taking $m = k$ yields $b = \mathbf{k}b \in \mathcal{T} \cap \mathcal{T}^\circ$, a contradiction.

If $a = b$ then we can cancel a from both sides of (6.1) to get $\mathbf{t} = \mathbf{u}$.

If $a = (-)b$ then Lemma 6.6(iii) says $(-)be' = b$, implying $b + b = (be)^\circ$ and thus $(-)b = b$, so we are back to the previous paragraph.

Thus, by Theorem 6.18 we may assume that $e' = \mathbf{1}$. If $a = (-)b$ then adding a to both sides of (6.1) yields $(\mathbf{t} + \mathbf{1})a = (\mathbf{u} - \mathbf{1})b$, and we are done by induction on u unless $u = 1$. But then $(\mathbf{t} + \mathbf{1})a = be$, and thus by Lemma 6.17, $\mathbf{k}a = (-)a$, for some $k \leq t$, implying $\mathbf{k} = (-)\mathbf{1}$, and thus $\mathbf{k} + \mathbf{2} = e' = \mathbf{1}$.

Thus we may assume that $a \neq (-)b$, and we add a to both sides of (6.1). The right side becomes $(a + b) + (\mathbf{u} - \mathbf{1})b$. If $a + b \neq (-)b$ then its height is $< u$ and we are done by induction on u . Hence, we may assume that $a + b = (-)b$.

Adding a to both sides of (6.1) yields $(\mathbf{t} + \mathbf{1})a = (\mathbf{u} - \mathbf{1})b$, and again we are done by induction on u unless $u \leq 2$. If $u = 2$ then $(\mathbf{t} + \mathbf{1})a = be$, and thus some $(\mathbf{k})a = (-)a$, implying $(-)a + b = (-)b$, and $a(-)b = b$. Hence $b + b = a + b^\circ = (-)(b + b)$, implying $(-)$ is of the first kind, i.e., $a + b = b$, a case we already handled.

Thus we conclude that $u = 1$, and $c := \mathbf{t}a \in \mathcal{T}$. Then $c(-)a = (\mathbf{t} - \mathbf{1})a$, so we conclude by induction unless $c = a$ or $t - 1 = 1$, i.e., $t = 2$. \square

Here are some uniqueness results concerning uniformity.

Proposition 6.27. *Suppose $\mathbf{t}a = \mathbf{u}b$ in a meta-tangible triple of the first kind. Then either $a = b$, $a + b = a$, $a + b = b$, or the triple has characteristic 2 with $2a = 2b$, implying both $a + 2b = a$ and $b + 2a = b$.*

Proof. Assume that $a \neq b$, and $a + b \notin \{a, b\}$. Then $\mathbf{1} = e' = \mathbf{3}$. Hence we may assume that $u, t \leq 2$. If $u = 2$ and $t = 1$ then $a = 2b \in \mathcal{T} \cap \mathcal{T}^\circ$, contradiction, and likewise if $u = 1$ and $t = 2$. Hence $u = t = 2$. It follows that $a = 3a = a + 2b$. \square

Proposition 6.28. *Suppose $\mathbf{m}_1a + \mathbf{m}_2b = \mathbf{m}_3c$ for $a, b, c \in \mathcal{T}$ in a meta-tangible triple of the first kind, with $a \neq b$. Then $a = b$, $a = c$, $b = c$, $a + b = c$, or $a + b + c \in \{a, b, c\}$, or the triple has characteristic 2 with $2a + 2b = 2c$. In the latter case, $c + 2(a + b) = c$.*

Proof. Assume that $a \neq b$, and $\mathbf{m}_1 \geq \mathbf{m}_2$. If say $a + b = b$ we can eliminate m_1 , in which case Proposition 6.27 is applicable, and we need to check the case of characteristic 2. Again, by Proposition 6.27, $2a + 2b = 2b = 2c$.

If $a + b = a$ we make the symmetric argument.

Hence we may assume the triple is not $(-)$ -bipotent, and thus has characteristic 2. Hence we may reduce m_1, m_2, m_3 and have $m_1, m_2, m_3 \leq 2$. Assume that $\mathbf{m}_1 \geq \mathbf{m}_2 > 0$. Then

$$\mathbf{m}_3c = (\mathbf{m}_1 - \mathbf{m}_2)a + \mathbf{m}_2(a + b).$$

If $m_1 = m_2$ we can conclude again with Proposition 6.27

Hence we may assume that $m_1 = 2$ and $m_2 = 1$, i.e. $2a + b = c$. By uniqueness of the quasi-negative, $b = c$. \square

6.4. Surpassing relations on meta-tangible triples.

Let us now examine how surpassing relations can arise on meta-tangible triples to yield systems. Perhaps surprisingly, the same triple can support different surpassing relations, as to be seen in Example 6.31.

Theorem 6.29. *Suppose the triple $(\mathcal{A}, \mathcal{T}, -)$ is cancellative meta-tangible.*

- (i) \preceq_\circ is a surpassing relation, so $(\mathcal{A}, \mathcal{T}, -, \preceq_\circ)$ is a meta-tangible system.
- (ii) One of the following holds for a \mathcal{T} -surpassing relation \preceq :
 - (a) $\preceq = \preceq_\circ$;
 - (b) the triple is $(-)$ -bipotent of the first kind, of even characteristic (possibly 0). (We have called this the **exceptional** case; see Remark 6.30 for a more precise description.)

Proof. (i) We need to verify the conditions of Definition 1.45. Transitivity is clear, as are conditions (i) and (ii). Condition (iii) follows from Lemma 1.46. Condition (iv) follows from Proposition 5.2(iii), whereas (v) follows from Proposition 6.10. Condition (vi) is clear, since if $b = a^\circ + c^\circ = (a + c)^\circ$, then $b \in \mathcal{T} \cap \mathcal{A}^\circ$, a contradiction.

(ii) We say that a pair (a, b) with $a \preceq b$ is **usual** if $b = a + c^\circ$ for some c . We shall show that an unusual pair can occur only in the exceptional case.

If a or b has height ≥ 3 , Theorem 6.25 says that the system is exceptional. Thus we may assume that both have height ≤ 2 . We rely heavily on Lemmas 6.6 and 6.12. If $a, b \in \mathcal{T}$ then $a = b$ and the assertion is trivial. We cannot have a of height 2, by Corollary 6.7.

Thus we may assume that $b \notin \mathcal{T}$. $b = d^\circ$ for $d \in \mathcal{T}$. If $a + d = d$ then we are done (since then $a + d^\circ = d^\circ = b$), so we may assume by Proposition 5.2 that

$$a = a + d^\circ = a + b. \quad (6.2)$$

If $a \in \mathcal{T}^\circ$ then $b \preceq a$ and $a \preceq b$ implies $a = b$. So we may assume that $a \notin \mathcal{T}_0$. Hence $a = a_1^\circ$ for $a_1 \in \mathcal{T}$. If $a_1 + d = d$ then $a + d = (-)a_1 + d = d$, by Lemma 6.6, and we are done. Hence, we may assume that

$$a_1 = a_1 + d^\circ = a_1 + b. \quad (6.3)$$

If $e' = 1$ then $a + d \preceq b + d = d$, implying $a + d = d$, which we already handled. If $a_1 = \pm d$ then $e' = 1$ by (6.3), and again we are done. Hence $a_1 \neq \pm d$.

We just showed that $e' \neq 1$, implying that the system is $(-)$ -bipotent, and thus $a_1 + d = a_1$.

But now $a + d^\circ = a$, implying $b \preceq a$, and thus $b = a$ (since $a, b \in \mathcal{A}^\circ$) and $a_1 + a = a_1 + b = a_1$, implying $e' = 1$, which we already handled. \square

Namely, suppose $a + c^\circ = b$ for $b \in \mathcal{T}$. We need to conclude that $a = b$. This is clear by Proposition 5.2 if $a \in \mathcal{T}$. We cannot have this situation if a has height 2, since then $b \in \mathcal{A}^\circ$. Hence a has height ≥ 3 , and we may assume by Theorem 6.25 that the triple is exceptional. Write $a = \mathbf{m}a_1$ for $a_1 \in \mathcal{T}$. If m is even then $a \in \mathcal{A}^\circ$, again contradiction, so m is odd, and then $a = b$ by Proposition 5.2.

Remark 6.30. *We can continue the argument in the proof for the unusual pair of Theorem 6.29. Write $a = \mathbf{m}a_1$ and $b = \mathbf{m}'b_1$ for $a_1, b_1 \in \mathcal{T}$, i.e., we play with the relation*

$$\mathbf{m}a_1 \preceq \mathbf{m}'b_1.$$

Clearly $m' \geq 2$, since otherwise $b \in \mathcal{T}$, implying $a = b$.

If $a_1 + b_1 = a_1$, we add b_1 on both sides to get $a_1 + b_1^\circ = a_1$, and thus $a_1^\circ + b_1^\circ = a_1^\circ$, implying $b_1^\circ \preceq a_1^\circ$ and thus $b_1^\circ = a_1^\circ$. Hence

$$a_1 = a_1 + b_1^\circ = a_1 + a_1^\circ = \mathbf{3}a_1,$$

contrary to $\mathbf{3} \neq 1$. Thus we may assume that $a_1 = b_1$ or $a_1 + b_1 = b_1$.

We also can squeeze out information about how m and m' relate to the characteristic and cycling (Example 5.11).

If $a_1 = b_1$, then $a \preceq_\circ b$ if $m' - m$ is even, so the exceptional case requires $m' - m$ to be odd. (Conceivably $m > m'$, but we could increase m' by adding on an even number of b_1 .)

If $a + b = b$, then $a \preceq_\circ b$ if m' is even, so the exceptional case requires m' to be odd.

Suppose that \mathcal{A} has characteristic k . In either case, if k were odd, we would replace m' by $m' + k$ if necessary to get a contradiction.

The exceptional case of Theorem 6.29 does arise in tropical mathematics, albeit rarely:

Example 6.31.

Take the \mathbb{N} -layered system of Example 1.29(i) (of characteristic 0), with $L = \mathbb{N}$.

- (i) Write $(\ell_1, a_1) \preceq (\ell_2, a_2)$ if $a_1 < a_2$, or if $a_1 = a_2$ with $\ell_1 \leq \ell_2$. Here $\mathbf{1} \preceq \mathbf{1} + \mathbf{1} = \mathbf{2}$ but $\mathbf{1} \not\preceq_{\circ} \mathbf{2}$.
- (ii) Write $(\ell_1, a_1) \preceq (\ell_2, a_2)$ if $a_1 < a_2$ with $\ell_2 > 1$, or if $a_1 = a_2$ with $\ell_1 < \ell_2$. Here $\mathbf{2} \preceq \mathbf{3}$ but $\mathbf{2} \not\preceq_{\circ} \mathbf{3}$.

One also has unusual pairs arising from Example 5.11.

Theorem 6.29 enables us to incorporate \preceq_{\circ} into the system of a meta-tangible triple, but also we can make use of the hyperfield surpassing relation when desired.

6.5. Most meta-tangible systems are matroidal and \mathcal{T} -reversible.

The next lemma in conjunction with Theorem 6.25 “almost” says that meta-tangible systems are matroidal.

Lemma 6.32. *If $a + \mathbf{m}b \in \mathcal{A}^{\circ}$ for $a, b \in \mathcal{T}$ in a meta-tangible system, with $a \neq (-)b$, and m minimal such, then $a \preceq (-)\mathbf{m}b$; in fact, $a + (\mathbf{m} - \mathbf{1})b^{\circ} = (-)\mathbf{m}b$. Furthermore, if $e' \neq \mathbf{1}$, then $a + b = b$.*

Proof. By assumption, $m \geq 2$, and $a + \mathbf{m}'b \in \mathcal{T}$ for all $m' < m$. Hence $a + (\mathbf{m} - \mathbf{1})b = (-)b$, implying

$$a + (\mathbf{m} - \mathbf{1})b^{\circ} = (-)b(-)(\mathbf{m} - \mathbf{1})b = (-)\mathbf{m}b,$$

implying $a \preceq (-)\mathbf{m}b$.

If $a + b = a$ then $(-)b = a + \mathbf{m}'b = (a + b) + b + \cdots + b = a$, contrary to hypothesis.

If $e' \neq \mathbf{1}$, then the system is $(-)$ -bipotent, implying $a + b = b$. □

Remark 6.33. *Note in Lemma 6.32 that $m > 1$.*

Before giving the next result, we need to exclude a weird counterexample.

Example 6.34. *In the layered system of Example 1.29(vi), for $n = 9$, take $a = (1, \mathbf{1})$, and $c = \mathbf{6} = (6, \mathbf{1})$. Then $a + c^{\circ} = \mathbf{9} = c^{\circ}$, but we cannot write $a + d^{\circ} = c^{\circ}$ because the parities do not match.*

Here the effect of the characteristic comes too far up the line.

Thus the matroidal property can fail!

Proposition 6.35. *Every meta-tangible system of height 2 is matroidal.*

Proof. Suppose $a + c \in \mathcal{A}^{\circ}$, for $a \in \mathcal{T}$. Since c is uniform by Theorem 6.25, we are done in view of Lemma 6.32 unless c has height 1 or 2. If $c \in \mathcal{T}$ then $c = (-)a$ and we are done. The remaining case is $c = b(-)b$. If $a \neq b$ then $a(-)b \in \mathcal{T}$, implying $a(-)b = (-)b$ (since otherwise $a(-)b + b \in \mathcal{T} \cap \mathcal{T}^{\circ} = \{\emptyset\}$, implying $c = -a$, and we are done). Hence $a + b^{\circ} = b^{\circ} = c$. □

We have a similar but better story for reversibility.

Lemma 6.36. *If $a \preceq b + c$ for $a, b, c \in \mathcal{T}$, then $b \preceq a(-)c$.*

Proof. Write $a + d^{\circ} = b + c$. Then $(a + d)^{\circ} = b + c(-)a$. If $c \neq a$ then $c(-)a$ is tangible, and thus equals $(-)b$ (since otherwise $b + c(-)a \in \mathcal{T} \cap \mathcal{T}^{\circ} = \emptyset$, implying $(-)b = c(-)a$ anyway).

Thus we may assume $c = a$. If $b + c$ is tangible then $a = b + c$, implying $b \preceq b + c^{\circ} = (b + c)(-)c = a(-)c$. □

“Most” cancellative meta-tangible systems are \mathcal{T} -reversible, after another weird counterexample.

Example 6.37. *In the layered system of Example 1.29(vii), for $n = 5$, take $a = (1, \mathbf{1})$, $b = (1, \mathbf{2})$, and $c = (4, \mathbf{1})$. Then $a + \mathbf{3}^{\circ} = (5, \mathbf{2}) = b + c$, but we cannot write $b + d^{\circ} = a + c = c$ because the parities do not match.*

Here the effect of the characteristic comes too far up the line.

Theorem 6.38. *In a cancellative meta-tangible system, if $a \preceq b + c$ for $a, b \in \mathcal{T}$, then $b \preceq a(-)c$, except in the following situation (given in Example 6.37): There are $1 < m' \leq m$ such that $c = \mathbf{m}b$, $\mathbf{m}' = \mathbf{m}$ but $\mathbf{m}' - \mathbf{2} \neq \mathbf{m} - \mathbf{2}$, and $a + c = c$.*

Proof. If $b = a$, there is nothing to prove, so we assume that $b \neq a$.

We are given

$$a + d^\circ = b + c. \quad (6.4)$$

If $c \in \mathcal{T}$ then we are done by Lemma 6.36.

If $c \in \mathcal{A}^\circ$, then $a(-)b = b^\circ + c \in \mathcal{A}^\circ$, so $b = a$ and again we are done.

Thus, using Theorem 6.25, we may write $c = \mathbf{m}c_1$, for $c_1 \in \mathcal{T}$, and assume that $m \geq 2$.

First assume that $\mathbf{3} = \mathbf{1}$; then we can reduce to $m \leq 2$. If $(-)$ is of second kind then $b, c \in \mathcal{T}$ and again we are done by Lemma 6.36. Thus $(-)$ is of first kind, and $m = m' = 2$. Then

$$(a + d) + d \in \mathcal{A}^\circ,$$

implying $a + d = d$. Furthermore, $b \neq c_1$ since otherwise $a + d^\circ = \mathbf{3}b = b$, implying $a = b$ and we are done. If $b + c = b$ then again $a + d^\circ = b$, so $b + c_1 = c_1$ and $d^\circ = a + d^\circ = c_1^\circ = c$.

If $b + d^\circ = b$ then

$$a + b = a + b + d^\circ = b + b + c \in \mathcal{A}^\circ,$$

implying $a = b$ and we are done. If $b = d$ then

$$b^\circ = d^\circ = a + d^\circ = b + c,$$

and thus

$$b = b + b^\circ = b^\circ + c \in \mathcal{A}^\circ,$$

a contradiction. Hence we are left with $b + d = d$, so

$$b + d^\circ = d^\circ = a + d^\circ = a + (a + d^\circ) = c + a = c(-)a,$$

as desired.

Hence we may assume that $e' \neq \mathbf{1}$, so we have $(-)$ -bipotency.

We claim that it is enough to prove that

$$b + (\mathbf{m}' - \mathbf{1})(c')^\circ = a(-)c' \quad (6.5)$$

where $c' = \mathbf{m}'c_1$ for some $m' \leq m$.

First we conclude the proof modulo the claim, noting that $b(-)a \in \mathcal{T}$. If $b(-)a \neq (-)c_1$ then Lemma 6.32 (taking $b(-)a$, instead of a and c instead of b) yields $b(-)a \preceq (-)c'$, and thus $b + a^\circ \preceq a(-)c'$, implying $b \preceq a(-)c'$, and we are done. Hence we may assume that $b(-)a = (-)c_1$, and again we are done.

It remains to prove the claim, which is clear if $(-)$ is of second kind, for then $c' = c$ and we are done.

Thus, we may assume that $(-)$ is of the first kind. Then we are done if $m - m' = 2$, since we just add something from \mathcal{A}° from both sides. The remainder of the proof is via a case by case analysis using bipotency on b, c_1 , and d_1 . The idea is to reduce to the situation where they are all equal and surpass a , and then transfer c from one side to the other.

Write $d = \mathbf{m}''d_1$. Then $d^\circ = \mathbf{2m}''d_1$.

If $c_1 + d_1 = d_1$ then in (6.5) we can replace d_1 by $c_1 + d_1$ and increase m' by 1, so we are done. If $c_1 + d_1 = c_1$, then $a(-)c = a + d^\circ(-)c = b + c^\circ$ and we are done.

Hence $c_1 = d_1$.

If $a + c_1 = a$ then in (6.5) we can replace a by $a + c_1$ and increase m' by 1, so we are done. Hence either $a + c_1 = c_1$ or $a = c_1$.

If $b + c_1 = b$ then unique negation yields $b = a$, a contradiction. Hence either $b = c_1$ or $b + c_1 = c_1$.

Next assume that $b + c_1 = c_1$. If $a + c_1 = c_1$, then

$$b + d^\circ = d^\circ = a + d^\circ = b + c = a + c,$$

as desired. If $a = c_1$, then

$$b + d^\circ + a^\circ = d^\circ + a^\circ = a + a + d^\circ = a + b + c = a + c,$$

as desired.

Hence we may assume that $b = c_1$. If $a = c_1$, then $a = b$ and we are done.

Hence $a + c_1 = c_1$, and we are given $\mathbf{2m}''c_1 = (\mathbf{m} + \mathbf{1})c_1$, so $\mathbf{2m}'' = (\mathbf{m} + \mathbf{1})$. By hypothesis, $\mathbf{2(m}'' - \mathbf{1}) = \mathbf{m} - \mathbf{1}$, implying

$$b + \mathbf{2}(\mathbf{m}'' - \mathbf{1})c_1 = \mathbf{m}c_1 = c,$$

as desired. \square

6.6. \mathcal{T} -classical meta-tangible triples.

Definition 6.39. A semigroup $(\mathcal{A}, +)$ with tangible elements \mathcal{T} and negation map $(-)$ is **\mathcal{T} -classical** if $a^\circ = b^\circ$ for some $a \neq (\pm)b$ tangible.

$(\mathcal{A}, +)$ is **\mathcal{T} -nonclassical** if $a^\circ = b^\circ$ implies $a = (\pm)b$.

Note that a \mathcal{T} -nonclassical semigroup must be anti-negated (Definition 6.44) since $a(-)a = 0 = 0(-)0$ implies $a = 0$.

Lemma 6.40. A \mathcal{T} -invertible meta-tangible triple $(\mathcal{A}, \mathcal{T}, -, \preceq)$ is \mathcal{T} -classical iff $a^\circ = e$ for some $a = (\pm)\mathbf{1}$ in \mathcal{T} .

Proof. If $a^\circ = b^\circ$ for $a \neq (\pm)b$ tangible, then $(ab^{-1})^\circ = e$. \square

(The usual classical triple satisfies $a^\circ = b^\circ = 0$ for all a, b .)

Corollary 6.41. If a cancellative meta-tangible triple $(\mathcal{A}, \mathcal{T}, -, \preceq)$ is \mathcal{T} -classical, then $e' = \mathbf{1}$.

Proof. Suppose $a^\circ = b^\circ$ with $a \neq (\pm)b$. If $e' \neq \mathbf{1}$ then by $(-)$ -bipotence we may assume that $a + b = b$. But then by Lemma 6.12,

$$b = a + b = a + (b(-)a) = a^\circ + b = b^\circ + b = e'b,$$

implying $e' = \mathbf{1}$ after all. \square

There is a nice partial converse.

Lemma 6.42. Any non- $(-)$ -bipotent meta-tangible triple $(\mathcal{A}, \mathcal{T}, -, \preceq)$ is \mathcal{T} -classical.

Proof. Suppose $a^\circ = b^\circ$ with $a \neq (\pm)b$ and $a + b \neq a, b$. Then $a^\circ + b = (a + b)(-)a \in \mathcal{T}$, implying $a^\circ + b = b$ by Proposition 5.2(ii). Hence $(a + b)^\circ = a^\circ + b(-)b = b(-)b = b^\circ$, with $a + b \neq b$. \square

Here is a way of “eliminating” the classical part of a meta-tangible system, continuing Proposition 4.10.

Proposition 6.43. Any metasystem $(\mathcal{A}, \mathcal{T}, (-), \preceq)$ has the congruence $\Phi = \{(a_1, a_2) : a_1^\circ = a_2^\circ\}$, and the system of \mathcal{A}/Φ is $(-)$ -bipotent of first kind.

Proof. If $(a_1, a_2), (a'_1, a'_2) \in \Phi$ then

$$(a_1 a'_1)^\circ = a_1^\circ a'_1^\circ = a_2^\circ a'_2^\circ; \quad (a_1 + a'_1)^\circ = a_1^\circ + a'_1^\circ = a_2^\circ + a'_2^\circ,$$

implying Φ is a congruence, modulo which $(-)a$ becomes a since $a^\circ = ((-)a)^\circ$. Furthermore, Lemma 6.6 yields $(-)$ -bipotence since $a^\circ + b = b$ implies $(a + b)^\circ = a^\circ + b^\circ = b^\circ$. \square

6.6.1. Anti-negated triples.

Sums are rarely 0 .

Definition 6.44. We call a triple **anti-negated** if $a(-)a \neq 0$ unless $a = 0$.

It follows that if $a + b = 0$ for $a, b \in \mathcal{T}$ then $a = 0$. This property has different names in the literature: An **antiring** in [21, 73], “zero-sum free” in [30], and “lacking zero sums” in [44]. Every anti-classical triple is anti-negated, since $a(-)a = 0 = 0^\circ$ implies $a = 0$.

Lemma 6.45. In a meta-tangible triple $(\mathcal{A}, \mathcal{T}, (-))$, if some sum of nonzero tangible elements $\sum_{i=1}^t a_i$ is 0 , with $a_t \neq 0$ and $t \geq 2$ minimal, then one of the following holds:

- (i) $t = 2$ with $a_2 = -a_1$ (the classical negative).
- (ii) $t \geq 3$, $(-)$ is of the first kind, and $\mathbf{t} = 0$.

Proof. Otherwise, if $a_i \neq (-)a_j$, then we could replace $a_i + a_j$ by their tangible sum and reduce t . When $t \geq 2$ this means some $a_k = 0$, so we remove a_k and conclude by induction.

Thus, we may assume that all of the a_i are quasi-negatives of each other. If $t \geq 3$, then all of the a_i are equal with $(-)a_i = a_i$. Hence $a_t \mathbf{t} = \sum_{i=1}^t a_t \mathbf{1} = a_t 0$. Canceling a_t yields $(-)\mathbf{1} = \mathbf{1}$ and $\mathbf{t} = 0$.

We are left with the case $t = 2$, in which case $a_1 + a_2 = 0$, so $a_2 = -a_1$. \square

Lemma 6.46. *Any \mathcal{T} -ub meta-tangible triple is anti-negated, satisfies $e' \neq 1$, and is $(-)$ -bipotent.*

Proof. The first assertion is immediate. If $e' = 1$, the ub property implies $1(-)1 = 1 \in \mathcal{T}_0 \cap \mathcal{T}_0^\circ = \emptyset$, a contradiction. Hence \mathcal{T} is $(-)$ -bipotent by Theorem 6.18. \square

6.7. Squares and sums of squares.

We need some analog of the classical theory of real closed fields, in which the squares are always positive.

Lemma 6.47. *Suppose N is a subgroup of a group \mathcal{T} , containing all squares, which is maximal with respect to the property that $N \cap (-)N = \emptyset$. (Such N exists by Zorn's lemma.) Then for any $a \in \mathcal{T}$ we have t for which $a^{2^t} \in (-)N$ or $(-)a^{2^t} \in (-)N$.*

Proof. For $a \in \mathcal{T} \setminus N$, we could adjoin a to N unless $a^i b = (-)a^j b'$ for $b, b' \in N$. Then

$$a^{i-j} = (-)b'b^{-1} \in (-)N.$$

Take m minimal such that $a^m \in (-)N$, and write $m = 2^t q$ for q odd. Then replacing a by a^{2^t} , we may assume that $a^q \in (-)N$.

But $a^2 \in N$, so reducing the power q modulo 2 must yield 1, i.e., $a \in (-)N$. Likewise (taking a further power of 2) we could adjoin $(-)a$ to N unless $(-)a \in N$. By maximality of N , we have $a \in N$ or $-a \in N$ for each $a \in \mathcal{T}$, as desired. \square

Definition 6.48. *A monoid \mathcal{T} with negation map is **real** if $(-)a$ is not a square in \mathcal{T} , for each $a \in \mathcal{T}$. The triple $(\mathcal{A}, \mathcal{T}, (-))$ is **real** if \mathcal{T} is real.*

Lemma 6.49. *Suppose N is a subgroup of a real group \mathcal{T} , containing all squares, which is maximal with respect to the property that $N \cap (-)N = \emptyset$. (Such N exists by Zorn's lemma.) Then $\mathcal{T} = N \cup (-)N$.*

Proof. By Lemma 6.47, since $t = 1$. \square

Theorem 6.50. *Suppose N is a subgroup of a real group \mathcal{T} , containing all squares, which is maximal with respect to the property that $N \cap (-)N = \emptyset$. Then $\mathcal{T} = N \cup (-)N$. Furthermore, suppose that $(\mathcal{A}, \mathcal{T}, (-), \preceq)$ is an $(-)$ -bipotent triple, and let $(\mathcal{A}', +)$ be the semigroup generated by N . Then the map $\varphi : \mathcal{A} \mapsto (\mathcal{A}')_{\text{sym}}$ given by $a \mapsto (a, 0)$ and $(-)a \mapsto (0, a)$ for $a \in \mathcal{T}'$ is an isomorphism.*

Proof. Thus $\mathcal{T} = N \cup (-)N$, by Lemma 6.47, implying $\mathcal{A} = \mathcal{A}' \cup (-)\mathcal{A}'$. We write $a > b$ in \mathcal{T} if $a + b = a$. The map φ is a homomorphism, since for $a > b \in N$ we have

$$\varphi(a(-)a) = (a, a) = \varphi(a)(-)\varphi(a);$$

$$\varphi(a(\pm)b) = (a, 0) = \varphi(a)(\pm)\varphi(b);$$

$$\varphi(b(\pm)a) = (0, b) = \varphi(b)(\pm)\varphi(a).$$

\square

Thus, \mathcal{A}' can be viewed as the set of “positive” elements. We also have an analog of the classical theory of real closed fields, in which the sum of squares is always positive. First a Pythagorean-type property.

Lemma 6.51. *In a $(-)$ -bipotent triple, the sum and difference of squares of elements of \mathcal{T} are squares.*

Proof. By Remark 1.21, if $a^2 = b^2$ then

$$a^2 + b^2 = a^2 + a^2 = a^2(e + e) = (ae)^2,$$

$$a^2(-)b^2 = a^2(-)a^2 = a^2(e(-)e) = a^2(e + e) = (ae)^2,$$

whereas for $a^2 \neq b^2$ we may assume that $a + b = a$ and have $a^2 = (a(-)b)^2 = a^2(-)b^2 + (ab(-)ab)$, implying that $a^2 = (a(-)b)^2 = a^2(-)b^2$, and thus also $a^2 = (a(-)b)^2 = a^2 + b^2$, by Lemma 6.12. \square

6.8. Sign maps on \mathcal{T} .

The following system ties in with ordered structures having a negation map, as well as in the context of systems, where we assume that the set \mathcal{T} of tangible elements is a multiplicative group.

Example 6.52. The **sign system** \mathcal{S}_{sys} is $(\mathcal{A}_{\text{sys}}, \mathcal{T}_{\text{sys}}, (-), \preceq_{\circ})$, where $\mathcal{A}_{\text{sys}} = \{-1, 0, 1, \infty\}$ endowed with the obvious multiplication, and with idempotent addition also satisfying $a + 0 = 0 + a = a$, $a + \infty = \infty + a = \infty$, $-1 + 1 = \infty$.

This should be compared with the hyperfield of signs in [7], cf. Example 12.8.

Definition 6.53. A **sign map** on a monoid \mathcal{T} is a multiplicative homomorphism

$$\text{sgn} : \mathcal{T} \rightarrow (\{-1, 0, 1, \infty\}, \cdot)$$

The sign map is **strict** if $\text{sgn}(\mathcal{T}) \subseteq \{-1, 1\}$.

When \mathcal{T} has a negation map $(-)$, we assume furthermore that $\text{sgn}((-)a) = -\text{sgn}(a)$. In particular, $(-)$ is of the second kind.

$\mathcal{T}^+ := \text{sgn}^{-1}(1)$ is called the set of **positive elements** and $\mathcal{T}^- = \text{sgn}^{-1}(-1)$ is called the set of **negative elements**. $\text{sgn}^{-1}(0)$ is called the set of **neutral elements**. We can extend this to an order on \mathcal{A} precisely when $(-)1$ is not a sum of squares.

A **sign map** on a system $(\mathcal{A}, \mathcal{T}, (-), \preceq)$ is a morphism to \mathcal{S}_{sys} for which $\mathcal{T} \mapsto \mathcal{T}_{\text{sys}}$.

(This is very close to the minus sign used in [27, §3.1].) In the context of universal algebra, one would require the property for all operators other than negation:

$\text{sgn}(\omega(a_1, \dots, a_m)) = (-1)^j 0^k$, where j (resp. k) is the number of a_i such that $\text{sgn}(a_i) = -1$ (resp. $\text{sgn}(a_i) = 0$).

Let $\varepsilon = \text{sgn}(1)$. Then $\varepsilon^2 = \text{sgn}(1^2) = \text{sgn}(1) = \varepsilon$, implying $\varepsilon \in \{0, 1\}$. If $\varepsilon = 0$ then $\text{sgn} = 0$, the trivial map. So we assume from now on that $\varepsilon = 1$.

Note that $\text{sgn}((-)1) = -1$. This means that $(-)1$ cannot be a square in \mathcal{T} , since if $(-)1 = a^2$ then $-1 = \text{sgn}((-)1) = \text{sgn}(a)^2$, which is impossible.

Then \mathcal{T}^+ is a submonoid of \mathcal{T} , with $\mathcal{T}^+ \cup \mathcal{T}^- = \mathcal{T}$ and $\mathcal{T}^+ \cap \mathcal{T}^- = \emptyset$.

Example 6.54. (i) \mathbb{R} has the classical sign map.

(ii) The semiring \mathcal{A} of Example 1.33 has a sign map, given by

$$\text{sgn}(0, 0) = 0, \quad \text{sgn}(a, 0) = 1, \quad \text{sgn}(0, a) = -1, \quad \text{sgn}(a, a) = \infty, \quad \forall a \in \mathcal{G}.$$

$\mathcal{A}^+ = \mathcal{G} \times 0$. The monoid $\mathcal{A}^+ \cup \mathcal{A}^-$ is real. Indeed, suppose that

$$(-)1 = (0, 1) = (a_0, a_1)^2 = (a_0^2 + a_1^2, a_0 a_1 + a_1 a_0).$$

If $a_0 = 0$ or $a_1 = 0$ the second component is 0, and if $a_0 = a_1$ then both components are equal, contradictions.

Conversely, we have:

Lemma 6.55. Suppose N is a submonoid of a real group \mathcal{T} , containing all squares, which is maximal with respect to the property that $N \cap (-)N = \emptyset$. Then there is a sign map sgn on \mathcal{T} such that $N = \mathcal{T}^+$.

Proof. Using Lemma 6.47, we define $\text{sgn}(a) = 1$ iff $a \in N$. □

Proposition 6.56. Suppose $(\mathcal{A}, \mathcal{T}, (-), \preceq)$ is a meta-tangible system, with \mathcal{T} a real group. Then there is a sign map sgn on \mathcal{A} given by Lemma 6.55 on \mathcal{T} , and $\text{sgn}(0) = 0$, and $\text{sgn}(a^\circ) = \infty$ for each $a \neq 0$.

Proof. Take the sign map of Lemma 6.55, formally defining $\text{sgn}(a^\circ) = \infty$. □

The negation of a positive element will be negative, and visa versa. Any square a^2 is positive.

6.9. Classifying metasystems.

We have located all of the main examples of meta-tangible systems, which mainly lie within the tropical framework. Recall that $\mathbf{2} = \mathbf{1}$ exactly when \mathcal{A} is bipotent. In other words, characteristic 1 of the first kind gives the max-plus algebra, but it is not uniquely negated since $\mathcal{T} = \mathcal{T}^\circ$. The following result shows how systems naturally lead us to the other main tropical structures.

Theorem 6.57. *Any \mathcal{T} -invertible meta-tangible system $(\mathcal{A}, \mathcal{T}, (-), \preceq)$ over a semiring \mathcal{A} must satisfy one of the following:*

- (1) $(-)$ is of the first kind. $\mathcal{A} = \cup_{m \in \mathbf{N}} \mathbf{m}\mathcal{T}$, and $e' = \mathbf{3}$.
 - (a) $\mathbf{3} \neq \mathbf{1}$. Then \mathcal{T} is $(-)$ -bipotent, and $(\mathcal{A}, \mathcal{T}, (-), \preceq)$ is isomorphic to a layered system (either layered by \mathbf{N} or cycling in characteristic 0 (Example 5.11), and layered by \mathbb{Z}/k in characteristic $k > 0$).
In particular, when $\mathbf{3} = \mathbf{2}$, we have $\mathbf{m} = \mathbf{1}^\circ$ for all $\mathbf{m} \geq 2$, and $\mathcal{A} = \mathcal{T}^+$.
 - (b) $\mathbf{3} = \mathbf{1}$. Hence $(\mathcal{A}, \mathcal{T}, -, \preceq)$ has characteristic 2 and height 2. The semiring[†] \mathcal{A}° is bipotent, and the conclusions of Proposition 6.10 must hold.
- (2) $(-)$ is of the second kind. There are two possibilities:
 - (a) \mathcal{T} is $(-)$ -bipotent, and \mathcal{T} (and thus \mathcal{A}) is idempotent. Taking the congruence Φ as in Proposition 4.10, \mathcal{A}/\equiv is a $(-)$ -bipotent system of the first kind, under the induced addition and multiplication. When not exceptional, $\mathcal{A} = \mathcal{T}^+$. When real, \mathcal{A} is isomorphic to a symmetrized system
 - (b) \mathcal{T} is not $(-)$ -bipotent. Then the system is \mathcal{T} -classical, and the semiring[†] \mathcal{A}° is bipotent. Furthermore $e' = \mathbf{1}$. Hence $\mathcal{A} = \mathcal{T}^+$. Either $\mathbf{N} \subseteq \mathcal{T}$, or $(\mathcal{A}, \mathcal{T}, -, \preceq)$ has characteristic k for some $k \geq 1$. In the latter case, $(\mathcal{A}, \mathcal{T}, -, \preceq)$ is layered by \mathbb{Z}/k .

Proof. We start with Theorem 6.18, which says that \mathcal{T} is $(-)$ -bipotent or $e' = \mathbf{1}$. This enables us to subdivide parts (1) and (2) (although in the reverse order). Also, by Theorem 6.25, every element of \mathcal{A} is uniform.

(1) If $(-)$ is of the first kind, this means that $(-)a = a$ and all elements have the form $\mathbf{m}a$ for $a \in \mathcal{T}$. If \mathcal{T} is $(-)$ -bipotent, and $a + b = b$, we get

$$\mathbf{m}a + \mathbf{m}'b = \mathbf{m}'a,$$

$$\mathbf{m}a + \mathbf{m}'a = (m + m')b,$$

$$(\mathbf{m}a)(\mathbf{m}'b) = \mathbf{m}\mathbf{m}'ab,$$

which are precisely the rules for layered addition and multiplication, so \mathcal{A} is layered by \mathbf{N} . Eventually the numbers \mathbf{m} may cycle modulo k , in which case one can identify subsequent layers modulo k .

When $\mathbf{3} = \mathbf{2}$, we clearly have $\mathbf{m} = \mathbf{1}^\circ$ for all $\mathbf{m} \geq 2$. The last assertion is by Theorem 6.25.

When $\mathbf{3} = \mathbf{1}$, every element has height ≤ 2 by Theorem 6.25, and we conclude with Proposition 6.10, noting that $a^\circ + a^\circ = e'a + a = a + a = a^\circ$.

(2) First assume that \mathcal{T} is $(-)$ -bipotent, so \mathcal{T} (and thus \mathcal{A}) is idempotent. In particular, $\mathcal{A} = \mathcal{T}^+$, \mathcal{A}/\equiv is a $(-)$ -bipotent system by Lemma 6.23, and $(-)[a] = [(-)a] = [a]$.

By Theorem 6.50, if \mathcal{T} is real, we take a subgroup \mathcal{T}' of \mathcal{T} maximal with respect to $\mathcal{T}' \cap (-)\mathcal{T}' = \emptyset$. Letting \mathcal{A}' be the sub-semigroup of $(\mathcal{A}', +)$ generated by \mathcal{T}' , we see that the map $\mathcal{A} \mapsto (\mathcal{A}')_{\text{sym}}$ is an isomorphism, under the map $a \mapsto (a, 0)$ and $(-)a \mapsto (0, a)$ for $a \in \mathcal{T}'$. More generally, we apply Lemma 6.47.

Now assume that \mathcal{T} is not $(-)$ -bipotent, so $e' = \mathbf{1}$ by Theorem 6.18. The system is \mathcal{T} -classical by Lemma 6.42. Again Proposition 6.10 shows that \mathcal{A}° is bipotent, noting that $a^\circ + a^\circ = e'a + a = a + a = a^\circ$. By Proposition 6.16, either $\mathbf{N} \subseteq \mathcal{T}$ or $(\mathcal{A}, \mathcal{T}, -, \preceq)$ has characteristic k for some $k \geq 1$. In the latter case, $\mathbf{1}, \dots, \mathbf{k} - \mathbf{1}$, are distinct, since if $\mathbf{m} = \mathbf{m}'$ for $1 \leq m < m' \leq k$, adding $(-)\mathbf{m}' - \mathbf{1}$ to both sides lowers the characteristic, a contradiction. \square

6.10. Important examples of meta-tangible systems. Let us see how the important tropical examples fit into the classification of Theorem 6.57, which boils down mostly to the classical, supertropical, symmetrized, and layered examples.

- When $\mathbf{2} = \mathbf{3}$, Case (1a) boils down to the supertropical domain $\mathcal{A} = \mathcal{T}^+$ of height 2, where $\mathcal{A}^\circ = \mathcal{T}^\circ = \mathbf{2}\mathcal{T}$. We get the system $(\mathcal{A}, \mathcal{T}, (-), \preceq)$ of the first kind, where \mathcal{T} is the monoid of “tangible elements,” $(-)$ is the identity map, and \succeq is “ghost surpasses.” (Thus $a^\circ = a^\nu$.) Proof: For a, b tangible, $a + b$ is a ghost only when $b = a = (-a)$.
- In general, case (1a) becomes the layered structure, as stated in the theorem. Note that although the \mathbb{N} -layered system is $(-)$ -bipotent of first kind, \mathcal{A}° is not bipotent since $e + e = \mathbf{4} \neq \mathbf{2} = e$.
- In Case (1b), Lemma 6.6 says that we have the following possibilities for $a + b$:
 - (i) a° , iff $a = b$,
 - (ii) b ,
 - (iii) $c \in \mathcal{T}$, where $c + a = a^\circ + b = b$.

The classical algebra of characteristic 2 fits into (1b), with each $a^\circ = \mathbb{0}$, and one also has Example 1.34.

There also is the layered algebra of Example 1.29(v), whose system is meta-tangible of first kind but not $(-)$ -bipotent.

On the other hand, Proposition 6.43 enables one to mod out the classical part to reduce Case (1b) to the $(-)$ -bipotent case (1a).

- Case (2a) includes $(\mathcal{A}, \mathcal{T}, (-), \preceq)$ of Definition 1.27 (when of second kind) which is a $(-)$ -bipotent system, where \mathcal{A} is a supertropical domain, \mathcal{T} is the set of “tangible elements,” and \succeq is “ghost surpasses,” which is a \circ -surpassing relation. (Thus $a^\circ = a^\nu$. Same proof as before, using $(-)b$ instead of b . Indeed, if $a \preceq (-)b + c$ and $a = (-)b$, then $(-)b \preceq a + c$, implying $b \preceq (-)a(-)c$. Likewise, $a + b$ is a ghost only when $b = (-a)$.)
- “Layered semirings” (which come up in Cases (1a) and (2a)) were reviewed in Example 1.28. They are rather ubiquitous, although not always well-behaved (including the exceptional systems), especially when viewed slightly more generally.
 - (i) L can act as an index set on systems with a negation map:

Example 6.58. Suppose that $(\mathcal{A}, \mathcal{T}, (-), \preceq)$ is a \mathcal{T} -nonclassical $(-)$ -bipotent system, with $\mathcal{A} = \mathcal{T}^+$. For $a_1 \neq (-)a_2$, we write $a_1 > a_2$ when $a_1 + a_2 = a_1$.)

We assume that the “layering semiring” L also has a negation map that we also designate by $(-)$. The two natural examples are \mathbb{N} with the identity, and \mathbb{Z} with the usual negation.

We can define the **layered system** $\tilde{\mathcal{A}} = L \times \mathcal{A}$, where $\tilde{\mathcal{T}} = \{(\ell, a) \in L \times \mathcal{A} : \ell = (\pm)1\}$.

We define addition for $\tilde{\mathcal{T}}^+$ by

$$(\ell_1, a_1^\circ) + (\ell_2, a_2) = \begin{cases} (\ell_1, a_1) & \text{if } a_1 > a_2; \\ (\ell_2, a_2) & \text{if } a_1 < a_2; \\ (\ell_1 + \ell_2, \mathbf{2}a_1) & \text{if } a_2 = a_1; \\ (\ell_1 + \ell_2, a_1^\circ) & \text{if } a_2 = (-)a_1; \end{cases}$$

$$(\ell_1, a_1^\circ) + (\ell_2, a_2^\circ) = \begin{cases} (\ell_1, (a_1 + a_2)^\circ) & \text{if } a_1 \neq (\pm)a_2; \\ (\ell_1, a_1^\circ) & \text{if } a_1 = (\pm)a_2. \end{cases}$$

We define $(-)(\ell, a) = ((-)\ell, (-)a)$. Thus $(\ell, a) \in \tilde{\mathcal{T}}$ iff $\ell = (\pm)1$.

We also can obtain layered systems by symmetrizing L ; namely we take $\hat{L} = L \times L$ with negation being the switch map $(\ell_1, \ell_2) \mapsto (\ell_2, \ell_1)$. Note that applied to (ii), \mathcal{T} is a subsystem isomorphic to the example of [3].

- (ii) This context includes Parker’s exploded tropical structure, studied axiomatically under the name of ELT-algebra by Sheiner [70]. They are of the form $L \times \mathcal{G}$, where L is the ring of leading coefficients upstairs in the pre-tropicalized world of Puiseux series. Explicitly, writing (ℓ, a) for the element a in layer ℓ , we have multiplication

$$(\ell_1, a_1)(\ell_2, a_2) = (\ell_1\ell_2, a_1a_2)$$

and addition

$$(\ell_1, a_1) + (\ell_2, a_2) = \begin{cases} (\ell_1, a_1) & \text{for } a_1 > a_2, \\ (\ell_1 + \ell_2, a_1) & \text{for } a_1 = a_2. \end{cases}$$

The ensuing ELT-linear algebra has been studied by Blachar [11, 12] and Blachar-Sheiner [13]. One of the original motivations of this paper was to see whether the results of [11, 70] can be obtained in the more general setting of systems with negation.

- (iii) Another possibility is to take $L = \mathcal{A}$. Then one can define a different negation map $(-)(\ell, a) = (a, \ell)$, the switch, to get the symmetrized systems, but these are not meta-tangible and they fail to satisfy many of the conclusions of the theorems given here. For example,

$$(2, 0) \preceq (2, 0) + (1, 0), \quad \text{but} \quad (2, 0) \not\preceq (2, 0)(-)(1, 0) = (2, 0) + (0, 1) = (2, 1).$$

- Case (2a) leads to the following approach. If \mathcal{T} is real, then $(\mathcal{A}, \mathcal{T}, (-), \preceq)$ is isomorphic to the symmetrized system $(\mathcal{A}')_{\text{sym}}$ of Theorem 6.50. (This is layered over \mathcal{A} .) In general, \mathcal{A} is contained in a chain of quadratic extensions of a real symmetrized system. Akian and Gaubert are studying this sort of example.
- For Case (2b), in characteristic $\neq 2$, the classical system $(\mathcal{A}, \mathcal{A}, -, =)$ is of second kind, satisfying $e' = 1$. As stated in the theorem, all such systems in this case are classical, but there are some strange examples where $e' \neq 1$.

Ironically the theories diverge for first and second kinds, as we shall see when discussing linear algebra below.

6.11. Meta-tangible systems versus meta-tangible hypergroups.

In Proposition 5.7 we embedded the theory of hypergroups into that of systems, and this turns out to be categorical. We can go the other direction for meta-tangible systems.

Proposition 6.59. *Any meta-tangible (resp. $(-)$ -bipotent) system $(\tilde{\mathcal{T}}, \mathcal{T}, (-), \subseteq)$ gives rise to a $(-)$ -closed (resp. $(-)$ -bipotent) hypergroup structure on the set \mathcal{T} , as follows:*

Define $[a] = \{a' \in \mathcal{T} : a' + a = a\}$.

Then define addition on \mathcal{T} by

$$a \boxplus b = \begin{cases} a + b : & a \neq (-)b, \\ [a] : & a = (-)b. \end{cases}$$

Proof. We verify the conditions of Definitions 1.35 and 1.36.

Recall from Lemma 6.12 that $a \in [a]$ iff $a' + (-)a = (-)a$, so $[a] = [(-)a]$. Hence $a \boxplus (-a) = [a] = (-a) \boxplus a$, implying addition is commutative.

Next note that

$$[a] + b = \begin{cases} b & \text{if } b > a; \\ [a] & \text{if } b = a \\ [b] \cup (b, a] = [a] & \text{if } b < a. \end{cases}$$

We need to check associativity. $(a_1 \boxplus a_2) \boxplus a_3 = a_1 \boxplus (a_2 \boxplus a_3)$ is clear unless one of the following holds:

- (i) $a_1 = (-)a_2$,
- (ii) $a_2 = (-)a_3$,
- (iii) $a_1 + a_2 = (-)a_3$,
- (iv) $(-)a_1 = a_2 + a_3$,

which we check respectively. But note that in each case the end result is to apply the brackets to each side whenever one encounters equality.

- (i) If $a_1 + a_3 = a_3$ then also $a_2 + a_3 = a_3$ and $(a_1 \boxplus a_2) \boxplus a_3 = [a_1] + a_3 = a_1 \boxplus (a_2 \boxplus a_3)$.
If $a_1 + a_3 = a_1$ then also $a_2 + a_3 = a_2$ and $(a_1 \boxplus a_2) \boxplus a_3 = [a_1] + a_3 = [a_1] = a_1 \boxplus a_2 = a_1 \boxplus (a_2 \boxplus a_3)$.
- (ii) Symmetric argument to (i).
- (iii) Suppose $a_1 \neq (-)a_2$ and $a_1 + a_2 = (-)a_3$. Then, by bipotence, $a_1 = (-)a_3$ or $a_2 = (-)a_3$, so $(a_1 \boxplus a_2) \boxplus a_3 = [a_3]$ and $a_1 \boxplus (a_2 \boxplus a_3)$ is either $(-)a_3 + (a_2 + a_3) = (-)a_3 + a_3 = [a_3]$ or $a_1 + [a_3] = [a_3]$.
- (iv) Symmetric argument to (iii).

Define $-a = (-)a$. Then the quasi-zeroes are exactly the sets $[a]$, which are the hyperzeros, and $a_1 + a_2$ is a hyperzero precisely when $a_2 = -a_1$. \square

(When there is multiplication, it is defined pointwise.)

7. SYMMETRIZATION

The next step in our program is to formalize the process mentioned in §1.4.5 and §1.4.6 that provides a negation map. It can be viewed as a special case of the following notion utilized long ago by physicists in studying elementary particles.

7.1. Supermodules and super-semialgebras.

Definition 7.1. A \mathbb{Z}_2 -graded semigroup M is also called a **super-semigroup**, i.e., $M = M_0 \oplus M_1$ as semigroups. A **super-semiring** R is a semiring[†] such that $(R, +)$ is a super-semigroup, which also satisfies $R_0^2, R_1^2 \subseteq R_0$ and $R_0 R_1, R_1 R_0 \subseteq R_1$.

A **super-module** over a super-semiring R is a module which is a super-semigroup $M = M_0 \oplus M_1$, satisfying $R_0 M_0 + R_1 M_1 \subseteq M_0$ and $R_0 M_1 + R_1 M_0 \subseteq M_1$.

In the more general universal algebraic context, a **super- $(\Omega; \mathcal{I})$ -algebra** will be a \mathbb{Z}_2 graded $(\Omega; \mathcal{I})$ -algebra $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ satisfying $\omega(a_0, \dots, a_m) \in \mathcal{A}_\ell$ whenever $a_i \in \mathcal{A}_{\ell_i}$ and $\ell_1 + \dots + \ell_m \equiv \ell \pmod{2}$.

The elements of $\mathcal{A}_0 \cup \mathcal{A}_1$ are called **homogeneous**. \mathcal{A}_0 is called the **even** component and \mathcal{A}_1 is called the **odd** component.

In particular, a semialgebra \mathcal{A} is a **super-semialgebra** if $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ as modules, where

$$\mathcal{A}_0^2, \mathcal{A}_1^2 \subseteq \mathcal{A}_0, \quad \mathcal{A}_0 \mathcal{A}_1, \mathcal{A}_1 \mathcal{A}_0 \subseteq \mathcal{A}_1.$$

\mathcal{A} is **super-commutative** if $a_i a_j = (-)^{ij} a_j a_i$ whenever $a_i \in \mathcal{A}_i, a_j \in \mathcal{A}_j$.

\mathcal{A} is **super-anticommutative** if $a_i a_j = (-)^{ij+1} a_j a_i$ whenever $a_i \in \mathcal{A}_i, a_j \in \mathcal{A}_j$.

This is the standard usage of the term “super,” somewhat different from our definition of supertropical algebra espoused earlier, and underlies the “symmetrization” procedure. Namely, we take $\hat{\mathcal{A}} = \mathcal{A} \times \mathcal{A}$, viewed in universal algebras via componentwise operations, and put $\mathcal{A}_0 = \mathcal{A} \times \{0\}$ and $\mathcal{A}_1 = \{0\} \times \mathcal{A}$.

But we would like to introduce a negation map on $\hat{\mathcal{A}}$ even when lacking it in \mathcal{A} . In order to identify the second component as the negation of the first, we employ an idea utilized in [2, Example 4.11] and [8, 52] (but here for semigroups instead of semirings[†]) which again is based on the familiar construction of \mathbb{Z} from \mathbb{N} , via pairs of natural numbers that are formally negated by switching their order. The negation map is the switch, and thus $\mathcal{A}^\circ = \{(a, a) : a \in \mathcal{A}\}$.

7.2. Symmetrization of modules.

Let us interpret this for modules.

Remark 7.2. If M is an R -module over a semiring[†] R , then $\hat{M} = M \times M$ is also an R -module, under the diagonal action $r(a_0, a_1) = (ra_0, ra_1)$.

We also want an appropriate module structure for \hat{M} over the semiring[†] \hat{R} (Definition 1.31), under the following action, motivated by the calculation

$$(r_0(-)r_1)(a_0(-)a_1) = (r_0 a_0 + r_1 a_1)(-)(r_0 a_1 + r_1 a_0)$$

for any module with negation.

Definition 7.3. The **twist action** on $\hat{M} := M \times M$ over \hat{R} is given by

$$(r_0, r_1)(a_0, a_1) = (r_0 a_0 + r_1 a_1, r_0 a_1 + r_1 a_0), \quad r_i \in R, a_i \in M. \quad (7.1)$$

Remark 7.4. \hat{M} is naturally an \hat{R} -module under the twist action.

This is an instance of a supermodule; $\hat{M} = \hat{M}_0 \oplus \hat{M}_1$ where $\hat{M}_0 = M \times \{0_M\}$ and $\hat{M}_1 = \{0_M\} \times M$, which is a supermodule over $\hat{R} = \hat{R}_0 \oplus \hat{R}_1$.

Example 7.5. For any semigroup M , \hat{M} is naturally a module over $\hat{\mathbb{N}}$.

When M already has a negation map $(-)$, Henry [35, §4] defines an equivalence relation \sim on \widehat{M} given by $(a_0, a_1) \sim ((-a_1, (-a_0))$. This can be viewed as a special case of Proposition 4.10.

Proposition 7.6. *If M is a semigroup, then the twist induces an action on \widehat{M}/\sim .*

Proof.

$$(r_0, r_1)(a_0, a_1) = (r_0a_0 + r_1a_1, r_0a_1 + r_1a_0) \sim ((-)(r_0a_1 + r_1a_0), (-)(r_0a_0 + r_1a_1)) = (r_0, r_1)((-)(a_1, a_0)).$$

□

7.3. Symmetrization of semirings and \mathcal{T} -semirings.

Any semiring (or \mathcal{T} -semiring) R injects into \hat{R} via $r \mapsto (r, 0)$, and $(1, 0)$ is the unit element of \hat{R} .

Lemma 7.7. *Suppose that R is a semiring.*

- (i) *If R is commutative then \hat{R} is commutative.*
- (ii) *If R is ub (for example supertropical), then \hat{R} is ub.*

Proof. (i) $(a_0, a_1)(b_0, b_1) = (a_0b_0 + a_1b_1, a_0b_1 + a_1b_0) = (b_0, b_1)(a_0, a_1)$.

(ii) Check addition in each component.

□

There is another way of viewing the symmetrization of the free module.

Example 7.8. *The free module with negation map (Example 4.4) can be viewed as the symmetrization of the free module (without negation) $R^{(I)}$, where we identify e_i with $(e_i, 0)$ and $(-e_i)$ with $(0, e_i)$. This will be useful when we deal with semirings arising from tensor product constructions.*

7.4. Symmetrization in the language of universal algebra.

Let us view symmetrization in the language of universal algebra (and also not be tied down with associativity):

Lemma 7.9. *Any $(\Omega; \mathcal{I})$ -algebra \mathcal{A} with the structure of R -module can be embedded into an $(\Omega; \mathcal{I})$ -algebra $\hat{\mathcal{A}} = \mathcal{A} \times \mathcal{A}$ with componentwise addition, and scalar multiplication over \hat{R} given by the twist action of Definition 7.3, and linear operators*

$$\omega_{m,j}((a_{1,0}, a_{1,1}), \dots, (a_{m,0}, a_{m,1})) = \left(\sum_{\iota \text{ even}} \omega_{m,j}(a_{1,\ell_1}, \dots, a_{m,\ell_m}), \sum_{\iota \text{ odd}} \omega_{m,j}(a_{1,\ell_1}, \dots, a_{m,\ell_m}) \right),$$

where $\ell_i \in \{0, 1\}$ for each i , and ι is the number of indices ℓ_i that equal 1.

Definition 7.10. $\hat{\mathcal{A}}$ is called the **symmetrized (Ω, \mathcal{I}) -algebra** of \mathcal{A} .

We write $\mathcal{I} \triangleleft \mathcal{A}$ when $\omega_{m,j}(a_1, \dots, a, \dots, a_m) \in \mathcal{I}$ for all operators $\omega_{m,j}$, $a \in \mathcal{I}$, and $a_u \in \mathcal{A}$.

Proposition 7.11. $\hat{\mathcal{A}}^\circ \triangleleft \hat{\mathcal{A}}$.

Proof. $(a_0, a_1)(a, a) = (a_0a + a_1a, a_1a + a_0a) \in \hat{\mathcal{A}}^\circ$.

Furthermore, for any operator $\omega_{m,j}$,

$$\omega_{m,j}((a_{j,1}), \dots, (a, a), \dots, (a_{j,m})) = \left(\sum_{\iota \text{ even}} \omega_{m,j}(a_{1,k_1}, \dots, a, \dots, a_{m,k_m}), \sum_{\iota \text{ odd}} \omega_{m,j}(a_{1,\ell_1}, \dots, a, \dots, a_{m,\ell_m}) \right) \in \hat{\mathcal{A}}^\circ. \quad (7.2)$$

since the summands match up.

□

Lemma 7.12. *For any commutative semiring[†] R , we have $\hat{R}[\lambda] \cong \widehat{R[\lambda]}$, under the map*

$$\sum_{\mathbf{i}} (\alpha_{0,\mathbf{i}}, \alpha_{1,\mathbf{i}}) \lambda_0^{i_1} \cdots \lambda_n^{i_n} \mapsto \left(\sum_{\mathbf{i}} \alpha_{0,\mathbf{i}} \lambda_0^{i_1} \cdots \lambda_n^{i_n}, \sum_{\mathbf{i}} \alpha_{1,\mathbf{i}} \lambda_0^{i_1} \cdots \lambda_n^{i_n} \right).$$

Proof. Match components, and note that the map is 1:1 and onto.

□

Definition 7.13. We define the relation \preceq_\circ on \widehat{M} by saying:

$$(a_0, a_1) \preceq_\circ (b_0, b_1) \quad \text{iff} \quad b_i = a_i + c \text{ for some } c \in M, \quad i = 0, 1. \quad (7.3)$$

The symmetrization process can be viewed as a universal in the sense of [50]:

Proposition 7.14. Suppose \mathcal{A} and \mathcal{A}' are (Ω, \mathcal{I}) -algebras, with \mathcal{A}' having a negation map. Then for any semiring[†] homomorphism $\varphi : \mathcal{A} \rightarrow \mathcal{A}'$, there is a unique homomorphism $\hat{\varphi} : \widehat{\mathcal{A}} \rightarrow \mathcal{A}'$ satisfying $\hat{\varphi} \circ \psi = \varphi$, such that $\hat{\varphi}(a_0, a_1) = (-)\varphi(a_0, a_1)$.

Proof. Define $\hat{\varphi}(a_0, a_1) = \varphi(a_0)(-)\varphi(a_1)$, so that the negation matches the switch. The verifications (including $\hat{\varphi}(a_0, a_1) = (-)\hat{\varphi}(a_1, a_0)$) are straightforward. \square

Corollary 7.15. (pointed out by Weibel) The injection of \mathcal{A} into the first component of $\widehat{\mathcal{A}}$ is a retract of the signature with negation (where the negation map of \mathcal{A} is the identity), whose back map $\widehat{\mathcal{A}} \rightarrow \mathcal{A}$ is given by $(a_0, a_1) \mapsto a_0 + a_1$.

Proof. Take $\widehat{\mathcal{A}}$ and \mathcal{A} instead of \mathcal{A} and \mathcal{A}' respectively, and the negation map to be the identity. \square

7.4.1. Symmetrization with involution.

Proposition 7.16. If $(R, *)$ is a semialgebra with involution, then the symmetrized semialgebra \hat{R} also has an involution, given by

$$(r_0, r_1)^* = (r_0^*, r_1^*).$$

The symmetric elements are $\{(r_0, r_1) : \text{each } r_i \in R \text{ is symmetric}\}$. The antisymmetric elements are $\{(r, r^*) : r \in R\}$.

Proof. The first two assertions are seen by matching components. For the last assertion, $(-)(r_0, r_1) = (r_1, r_0)$, whereas $(r_0, r_1)^* = (r_0^*, r_1^*)$, so matching components shows that $r_1 = r_0^*$ in the antisymmetric case. \square

Corollary 7.17. The sets of symmetric and antisymmetric elements of \hat{R} are precisely $(\hat{R}, *)^+$ and $(\hat{R}, *)^-$ respectively. The set of elements of \hat{R} that are both symmetric and antisymmetric is precisely $((R, *)^+)^{\circ}$.

7.5. Modification of symmetrization.

Remark 7.18. We would prefer for the symmetrization of a meta-tangible system to be a meta-tangible system, but this does not hold in general. The difficulty is that in order to have the sum of tangible elements to be tangible, we need $(a, b) \in \hat{T}$ for each $a, b \in \mathcal{T}$. But then $(a, b) + (b', a) = (a + b', a + b) = (a, a)$ whenever b, b' are dominated by a , so we lose uniqueness of the negation. The way out of this dilemma would be to define $\mathcal{T}(\widehat{\mathcal{A}})$ to be instead $(\mathcal{T} \times \mathbb{0}) \cup (\mathbb{0} \times \mathcal{T})$, since the unique quasi-negative of $(a, \mathbb{0})$ is now $(\mathbb{0}, a)$. But now $\mathcal{T}(\widehat{\mathcal{A}})$ is not nearly closed under addition, so we lose meta-tangibility. The solution taken in [3], when \mathcal{T} is ordered, is to define a new addition, which we adapt for $\mathcal{T}(\widehat{\mathcal{A}})^+$:

$$\begin{aligned} (a_1, \mathbb{0}) + (\mathbb{0}, a_2) &= \begin{cases} (a_1, \mathbb{0}) & \text{if } a_1 > a_2; \\ (\mathbb{0}, a_1) & \text{if } a_1 < a_2; \\ (a_1, a_1) & \text{if } a_1 = a_2. \end{cases} \\ (a_1, a_1) + (\mathbb{0}, a_2) &= \begin{cases} (a_1, a_1) & \text{if } a_1 \geq a_2; \\ (\mathbb{0}, a_2) & \text{if } a_1 < a_2. \end{cases} \\ (a_1, a_1) + (a_2, a_2) &= \begin{cases} (a_1, a_1) & \text{if } a_1 > a_2; \\ (a_2, a_2) & \text{if } a_1 \leq a_2. \end{cases} \end{aligned}$$

$\mathcal{T}(\widehat{\mathcal{A}})^+$ now yields a meta-tangible system. This can be viewed in the context of layering, as seen in Examples 1.28 and 6.58.

7.6. The transfer principle.

This treatment essentially is a careful reformulation of [2, Corollary 4.18], expressed through universal algebra in order to increase its applicability.

The transfer principle, whose roots are in [65], was introduced in [26] and made explicit in [2]. It is based on a way of passing from semirings[†] to rings, by means of the symmetrization $\widehat{\mathbb{N}\{x\}}$ of the free semiring[†] $\mathbb{N}\{x\}$, with the switch negation map. When our signature has classical negation with $a - a = 0$, the universal relation $f = g$ simply states that $f - g$ is identically 0, or in other words $f = g$ in the corresponding free algebra.

We say that b **dominates** a if either $a = b$ or $a + b = b$; b **strictly dominates** a if b dominates a with $a \neq b$.

The following result, formulated in universal algebra, slightly tightens the hypothesis in [2, Theorem 4.21]. In order to pass back and forth from \mathbb{N} to \mathbb{Z} , we assume that all universal identities are \mathbb{N} torsion free, i.e., the natural map $\mathbb{N}\{x; \Omega, \mathcal{I}\} \rightarrow \mathbb{Z}\{x; \Omega, \mathcal{I}\}$ is injective. (This is obvious when $\mathcal{I} = 0$.)

Theorem 7.19 (Transfer principle, strong form). *Suppose that the semiring[†] $\widehat{\mathbb{N}\{x; \Omega, \mathcal{I}\}}$ (under the usual operations of \mathbb{N}) satisfies an identical relation $(P_1, Q_1) \equiv (P_2, Q_2)$ for $P_i, Q_i \in \mathbb{N}\{x; \Omega, \mathcal{I}\}$ where also $(P_1, Q_1) \geq (P_2, Q_2)$. Then $(P_1, Q_1) \succeq_o (P_2, Q_2)$ in $\widehat{\mathbb{N}_n\{x; \Omega, \mathcal{I}\}}$ for every n (cf. Remark 1.57).*

Proof. We are given the identical relation $(P_1, Q_1) \equiv (P_2, Q_2)$ in the algebra $\widehat{\mathbb{N}\{x; \Omega, \mathcal{I}\}}$ and thus $\mathbb{Z}\{x; \Omega, \mathcal{I}\}$. Hence $P_1 + Q_2 \equiv P_2 + Q_1$ is an identical relation of $\mathbb{Z}\{x; \Omega, \mathcal{I}\}$ and thus of $\mathbb{N}\{x; \Omega, \mathcal{I}\}$, and thus of every carrier in the signature. By hypothesis, we can write $P_1 = P_2 + f$ and $Q_1 = Q_2 + g$. Then

$$P_2 + Q_2 + f = P_1 + Q_2 \equiv P_2 + Q_1 = P_2 + Q_2 + g.$$

For every evaluation in which f dominates the left side we must have g evaluating to the same, so we conclude that $(P_1, Q_1) \equiv (P_2, Q_2) + (\mathbf{n}f, \mathbf{n}g)$ on the free $(\Omega; \mathcal{I})$ -algebra. \square

Remark 7.20. *Because of the ambiguity of \mathbf{n} , it is misleading to deal with identities over \mathbb{N} whose coefficients are not $(\pm)1$.*

8. CATEGORIES OF SYSTEMS

Let us view systems in categorical terms, for further guidance.

8.1. Categories with negation.

We need to put the negation map into the framework of categories.

Definition 8.1. *A category \mathcal{C} has a **negation (endo)functor** $(-)$ if for each morphism f in $\text{Hom}(A, B)$ we have $(-)f \in \text{Hom}(A, B)$ satisfying $(-)(fg) = ((-)f)g = f((-)g)$ for any two compatible morphisms f, g .*

Lemma 8.2. *Any category \mathcal{C} of universal algebras \mathcal{A} with negation has a negation functor, given by*

$$((-)f)(a) := (-)f(a), \quad a \in \mathcal{A}.$$

Proof. $((-)(fg))(a) = (-)(f(g(a))) = ((-)f)(g(a)) = f((-)g(a)) = f((-)g)(a)$. \square

Remark 8.3. *In this notation, $(f(-)f)(\mathcal{A}^\circ) \subseteq \mathcal{A}^\circ$.*

8.2. Morphisms of systems.

We can make systems of the same signature into a category, by having the objects being the systems. The question is how to define morphisms. The customary way in universal algebra is to define a homomorphism $\varphi : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ so as to preserve the signature, in the sense that

$$\varphi(\omega_{m,j}(x_{1,j}, \dots, x_{m,j})) = \omega_{m,j}(\varphi(x_{1,j}), \dots, \varphi(x_{m,j}))$$

is satisfied. However, in the context of systems, it seems preferable to have the following broader definition. (Either definition provides a workable theory in the continuation of this theory.)

Definition 8.4. *A \preceq -**morphism** $\varphi : (\mathcal{A}, \mathcal{T}, (-), \preceq) \rightarrow (\mathcal{A}', \mathcal{T}', (-)', \preceq')$ of systems of the same signature is a map $\varphi : \mathcal{A} \rightarrow \mathcal{A}'$ satisfying the properties:*

- (i) $\varphi(\mathcal{T}) \subset \mathcal{T}'$,

- (ii) $\varphi((-)a) = (-)'\varphi(a)$,
- (iii) $\varphi(\omega(a_1, \dots, a_m)) \preceq \omega(\varphi(a_1), \dots, \varphi(a_m))$, $\forall a_\ell \in \mathcal{A}$.
- (iv) If $a_\ell \preceq b_\ell$ for each ℓ , then

$$\varphi(\omega(a_1, \dots, a_m)) \preceq \omega(\varphi(b_1), \dots, \varphi(b_m)).$$

Remark 8.5. In universal algebra one recovers the more familiar **homomorphism** by replacing \preceq by $=$ in Definition 8.4(iii). At any rate, we recall that equality holds on all entries in \mathcal{T} .

Example 8.6. A \preceq -morphism of systems with negation map $(-)$ and addition and multiplication satisfies the following conditions for all $a_i \in \mathcal{A}_1$:

- (i) $\varphi((-)a_1) \preceq (-)\varphi(a_1)$;
- (ii) $\varphi(a_1 + a_2) \preceq \varphi(a_1) + \varphi(a_2)$;
- (iii) $\varphi(a_1 a_2) \preceq \varphi(a_1)\varphi(a_2)$ for all $a_j \in \mathcal{A}_1$.

These conditions arise naturally in the cases of hypergroups and Lie semialgebras. For example, the left multiplication map $\ell_r : M \rightarrow M$ is a homomorphism iff left multiplication by r distributes over M . But ℓ_r is a \preceq -morphism iff $r(a_0 + a_2) \preceq ra_0 + ra_2$ for each $a_i \in M$, which is precisely the weak version of distributivity described in [74, §4.1].

Lemma 8.7. Any homomorphism φ of systems is a \preceq -morphism.

Proof. We prove that $a_1 \preceq a_2$ implies $\varphi(a_1) \preceq' \varphi(a_2)$. Indeed, $a_2 = a_1 + c^\circ$ implies $\varphi(a_2) = \varphi(a_1) + \varphi(c)^\circ$. (The operators behave analogously.) \square

Although a \preceq -morphism need not be a homomorphism in the universal algebra sense, it is strong enough to provide a viable theory, and as we shall see, provides a valuable tool to analyze tropicalization. One might be concerned that \preceq is reversed by inverses and negation maps (which reverse the order), but in these cases \preceq becomes equality:

Lemma 8.8. A \preceq -morphism $\varphi : \mathcal{A} \rightarrow \mathcal{A}'$ is a homomorphism, and thus isomorphism, if there is a \preceq -morphism $\psi : \mathcal{A}' \rightarrow \mathcal{A}$ such that $\psi\varphi = 1_{\mathcal{A}}$ and $\varphi\psi = 1_{\mathcal{A}'}$.

Proof.

$$\begin{aligned} \omega_{m,j}(a_{1,j}, \dots, a_{m,j}) &= \psi\varphi(\omega_{m,j}(a_{1,j}, \dots, a_{m,j})) \\ &\preceq \psi(\omega_{m,j}(\varphi(a_{1,j}), \dots, \varphi(a_{m,j}))) \preceq \omega_{m,j}(a_{1,j}, \dots, a_{m,j}), \end{aligned} \tag{8.1}$$

implying equality at each stage, so apply ψ in the middle. \square

8.3. Embedding hypergroups into systems. In Definition 5.23 we presented the system of a hypergroup. This can be made more explicit using the formalism of categories.

Theorem 8.9. There is a faithful functor Ψ from the category of hypergroups (as defined in [51]) into the category of uniquely negated \mathcal{T} -reversible systems, whose morphisms are the \preceq -morphisms, sending a hypergroup \mathcal{T} to its system $(\tilde{\mathcal{T}}, \mathcal{T}, (-), \subseteq)$. Furthermore, the hypergroup \mathcal{T} is meta-tangible, resp. closed, iff its system $(\tilde{\mathcal{T}}, \mathcal{T}, (-), \subseteq)$ is meta-tangible, resp. $(-)$ -bipotent.

Proof. The first assertion is clear from Proposition 5.24, and the second from Lemma 6.59. \square

Thus, closed hypergroups can be studied in terms of §6.

8.4. Congruences and \mathcal{T} -ideals on systems.

This discussion is inspired by Jun [51, §2], in which an algebraic structure theory is developed on hyperrings, which we put now into the framework of uniquely negated systems. Although this can be done strictly in terms of universal algebra, we specialize to the case where \mathcal{A} is a \mathcal{T} -semiring, for simplicity.

The idea here is that morphisms should be defined in terms of congruences, whose structure relates to $\mathcal{A} \times \mathcal{A}$ but not \mathcal{A} ; nevertheless we can relate it to \mathcal{T} by strengthening the definition of ideal.

Definition 8.10. Suppose $(\mathcal{A}, \mathcal{T}, (-), \preceq)$ has an action $\mathcal{T} \times \mathcal{A} \rightarrow \mathcal{A}$. A \mathcal{T} -ideal \mathcal{I} of $(\mathcal{A}, \mathcal{T}, (-), \preceq)$ is a \mathcal{T} -ideal \mathcal{I} of \mathcal{A} as an $(\Omega; \mathcal{I})$ -algebra, satisfying the following conditions, where $\mathcal{I}_{\mathcal{T}} = \mathcal{T} \cap \mathcal{I}$:

- (i) $\mathcal{I}_{\mathcal{T}}$ additively spans \mathcal{I} .
- (ii) If $a \in \mathcal{T}$ and $v \in \mathcal{I}_{\mathcal{T}}$ then $av \in \mathcal{I}_{\mathcal{T}}$.

- (iii) If $a \preceq b + v$, for $v \in \mathcal{I}$, then there is $w \in \mathcal{I}_{\mathcal{T}}$ for which $a \preceq b + w$.
- (iv) If $a \in \mathcal{T}$, then $a^\circ \in \mathcal{I}$.
- (v) $\omega_m(a_1, \dots, b, \dots, a_m) \in \mathcal{I}_{\mathcal{T}}$ for all $a_k \in \mathcal{T}$ and $b \in \mathcal{I}_{\mathcal{T}}$, and every operator ω_m other than addition.

Remark 8.11. The definition implicitly includes the condition that $(-)\mathcal{I}_{\mathcal{T}} = \mathcal{I}_{\mathcal{T}}$, since $(-)a = ((-)\mathbb{1})a$.

\mathcal{I} always denotes a \mathcal{T} -ideal in what follows.

Lemma 8.12. If $a \in \mathcal{I}$ and $a \preceq_\circ b$, then $b \in \mathcal{I}$.

Proof. Just write $b = a + c^\circ$, noting that $c^\circ \in \mathcal{I}$. □

A **(systemic) congruence** is a congruence on \mathcal{A} generated by pairs $\{(a_1, a_2) : a_i \in \mathcal{T}\}$.

Note that [51, Equation 6] holds here on \mathcal{T} .

The first stab at defining an ideal of a congruence Φ might be to take $\{a(-)b : (a, b) \in \Phi\}$, which works in classical algebra but is too crude for the max-plus, since it fails to be closed under addition. But things work when we restrict our attention to \mathcal{T} -ideals.

Definition 8.13. Given a \mathcal{T} -ideal \mathcal{I} , define the congruence $\Phi_{\mathcal{I}}$ by $a \equiv b$ iff we can write $a = \sum_j a_j$ and $b = \sum_j b_j$ for $a_j, b_j \in \mathcal{T}_0$ such that $a_j \preceq b_j + v_j$ for $v_j \in \mathcal{I}_{\mathcal{T}}$, each j .

Given a congruence Φ , define \mathcal{I}_{Φ} to be the additive sub-semigroup of \mathcal{A} generated by all $c \in \mathcal{T}_0$ such that $c \preceq a(-)b$ for some $a, b \in \mathcal{T}_0$ with $(a, b) \in \Phi$.

Lemma 8.14. In a \mathcal{T} -reversible system, $a \equiv b$ (with respect to $\Phi_{\mathcal{I}}$) for $a, b \in \mathcal{T}$, iff $\mathcal{I}_{\mathcal{T}}$ contains an element v such that $v \preceq a(-)b$.

Proof. This is clear for $a = b$, so we assume that $a \neq b$.

(\Rightarrow) If $a \equiv b$ then $a \preceq b + v$, for $v \in \mathcal{I}_{\mathcal{T}}$, and then $v \preceq a(-)b$.

(\Leftarrow) If $v \in \mathcal{I}_{\mathcal{T}}$ with $v \preceq a(-)b$, then $a \preceq b + v$, and $b \preceq (-)(v(-)a) = a(-)v$. □

This result generalizes [51, Lemma 3.6].

Remark 8.15. In a \mathcal{T} -reversible system, Condition (ii) of Definition 8.10 implies $w \preceq a(-)b$. Likewise, in Definition 8.13, $a_j \preceq b_j + v_j$ implies $b_j \preceq a_j(-)v_j$.

Proposition 8.16. In a \mathcal{T} -reversible system, $\Phi_{\mathcal{I}}$ is a congruence for any ideal \mathcal{I} . For any congruence Φ , \mathcal{I}_{Φ} is a \mathcal{T} -ideal. Furthermore, $\Phi_{\mathcal{I}_{\Phi}} \supseteq \Phi$ and $\mathcal{I}_{\Phi_{\mathcal{I}}} = \mathcal{I}$.

Proof. To check that $\Phi_{\mathcal{I}}$ is a congruence, first note that $a \equiv a$ since $\mathbb{0} \in \mathcal{I}$, and symmetry follows from Remark 8.15 when we write $a = \sum a_j$ and $b = \sum b_j$. Transitivity follows from Lemma 1.46, and then applying Condition (iii) of Definition 8.10. The defining condition is closed under the given operations, since if $a \equiv b$ and $a' \equiv b'$ for $a, a', b, b' \in \mathcal{T}$, then we write

$$a = \sum a_i, \quad b = \sum b_i, \quad a' = \sum a'_j, \quad b' = \sum b'_j, \quad a_i \preceq b_i + v_i, \quad a'_j \preceq b'_j + v'_j,$$

and then note that $a + a' = \sum a_i + \sum a'_j \preceq \sum (b_i + v_i) + \sum (b'_j + v'_j)$, the desired decomposition.

$\Phi_{\mathcal{I}}$ is closed under the given operations, since if $a \equiv b$ and $a' \equiv b'$ then writing $a \preceq b + v$ and $a' \preceq b' + v'$ we have $(a + a') \preceq (b + b') + (v + v')$ and thus $(a + a') \preceq (b + b') + w$ for some w .

\mathcal{I}_{Φ} is closed under the given operations, since if $(a, b) \in \Phi$ and $d \in \mathcal{T}$, then $da(-)db = d(a(-)b)$, and $c \preceq a(-)b$ implies $dc \preceq d(a(-)b) = da(-)db$.

To prove that $\Phi_{\mathcal{I}_{\Phi}} \supseteq \Phi$, note that if $(a, b) \in \Phi$ for $a, b \in \mathcal{T}$, then some $v \in \mathcal{I}_{\Phi}$ satisfies $v \preceq a(-)b$, and then $a \preceq b + v$, implying $(a, b) \in \Phi_{\mathcal{I}_{\Phi}} \supseteq \Phi$.

Likewise, to prove that $\mathcal{I}_{\Phi_{\mathcal{I}}} \supseteq \mathcal{I}$, note that if $v \in \mathcal{I}_{\mathcal{T}}$ then $(v, \mathbb{0}) \in \Phi_{\mathcal{I}}$ and $v = v(-)\mathbb{0} \in \mathcal{I}_{\Phi_{\mathcal{I}}}$.

For the reverse inclusion, suppose that $v \in \mathcal{I}_{\Phi_{\mathcal{I}}}$ is tangible. Then $v \preceq a(-)b \in \mathcal{I}$, for tangible $a \equiv b$ in $\Phi_{\mathcal{I}}$. By Definition 8.10, $v \preceq w$ for some $w \in \mathcal{I}_{\mathcal{T}}$, implying $v = w$ by Definition 1.45(iv). □

Remark 8.17. The last inclusion $\Phi_{\mathcal{I}_{\Phi}} \subseteq \Phi$ holds iff Φ satisfies the condition that if $(a, b) \in \Phi$ and $a(-)b = a'(-)b'$, then $(a', b') \in \Phi$. This leads to the notion of the **closure** $\bar{\Phi}$ of a congruence Φ to be generated by

$$\{(a', b') \in \mathcal{T} \times \mathcal{T}, a(-)b = a'(-)b' \text{ for } (a, b) \in \Phi\}.$$

But even without this, we have a program to mod out \mathcal{T} by $\mathcal{I}_{\mathcal{T}}$, namely take the congruence $\Phi := \Phi_{\mathcal{I}}$, and pass to $(\mathcal{A}/\Phi, \mathcal{T}/(\mathcal{T} \times \mathcal{T} \cap \Phi), (-), \preceq)$ where $(-)[a] = [(-)a]$.

This next result generalizes [51, Lemma 3.6]:

Lemma 8.18. *The restriction of Φ to \mathcal{T} specializes in the case of the system of Definition 5.23 to [51, Equation (10)].*

Proof. Given $a_j \preceq b_j + \mathcal{I}$ and $b_j \preceq a_j + \mathcal{I}$, we have (in \mathcal{T} as a hypergroup) $a_j \in b_j + \mathcal{I}$ and $b_j \in a_j + \mathcal{I}$, i.e., $a_j + \mathcal{I} = b_j + \mathcal{I}$. \square

From this point of view, the analogs of [51, Propositions 3.6, 3.11, and 3.15] then are obtained as applications of basic facts in the theory of universal algebra.

8.5. Tensor products with a negation map, and their semialgebras.

The tensor product is a well-known process in general category theory, [33, 54, 55, 72], and has been studied in the context of **monoidal categories**. Here we need the tensor product of modules and semialgebras (with a negation map) over commutative semirings[†]. These are described in terms of congruences, as given for example in [55, Definition 3] or, in our notation, [56, §3].

Let us work in a signature of modules over a commutative associative semiring[†] C . If V has a negation map $(-)$ in its additive signature, then we can incorporate the negation map into the tensor product, defining a negation map on $V \otimes_C W$ by $(-)(v \otimes w) = ((-)v) \otimes w$. When W also has a negation map we define a **negated tensor product** by imposing the extra axiom

$$((-)v) \otimes w = v \otimes ((-)w).$$

(This is done by modding out by the congruence generated by all elements $((-)v \otimes w, v \otimes (-)w)$ to the congruence defining the tensor product, which fits into the universal algebra framework.) From now on, the notation $V \otimes W$ includes this negated tensor product stipulation, and C is understood. Then $V \otimes W$ has the negation map given by $(-)(v \otimes w) = (-)v \otimes w$. This has some immediate consequences.

Lemma 8.19.

- (i) $(v \otimes w)^\circ = v^\circ \otimes w = v \otimes w^\circ$.
- (ii) $v^\circ \otimes w^\circ = (v \otimes w)^{\circ\circ}$.

Proof. (i) $(v \otimes w)^\circ = v^\circ \otimes w = (v(-)v) \otimes w = (v \otimes w) + ((-)v \otimes w) = v \otimes w + (v \otimes ((-)w) = v \otimes w^\circ$.

(ii) $v^\circ \otimes w^\circ = (v \otimes w^\circ)^\circ = (v^\circ \otimes w)^\circ = (v \otimes w)^{\circ\circ}$. \square

Remark 8.20. *One can easily prove the following facts, modifying say [67, Chapter 18]:*

- (i) *If $f_i V_i \rightarrow W_i$ are module morphisms then there is a unique map $f_1 \otimes f_2 : V_1 \otimes V_2 \rightarrow W_1 \otimes W_2$ satisfying*

$$(f_1 \otimes f_2)(v_1 \otimes v_2) = f_1(v_1) \otimes f_2(v_2).$$

- (ii) *The tensor product $(\mathcal{A}, \mathcal{T}, (-)) \otimes (\mathcal{A}', \mathcal{T}', (-)')$ of triples is a triple*

$$(\mathcal{A} \otimes \mathcal{A}', \{a_1 \otimes a'_2 : a_1 \in \mathcal{T}, a'_2 \in \mathcal{T}'\}, (-) \otimes 1_{\mathcal{A}'}).$$

- *This definition is suited towards “multilinear” algebra, since \mathcal{T} is the set of rank 1 tensors, together with $\mathbb{0}$.*
- *If the signature is closed under linearization, and F is a commutative associative semialgebra over R , then $_ \otimes_C F$ yields a functor from $(\Omega; \mathcal{I})$ -algebras over C to $(\Omega; \mathcal{I})$ -algebras over F . (In particular, this holds when F is the symmetrization of C .)*

Thus, modules with negated tensor products yield a monoidal category, and our discussion fits into this well-known theory. The theory runs most smoothly for free modules; when \mathcal{T} and \mathcal{T}' are free modules over the base semifield then the tensor product of uniquely negated triples is uniquely negated.

Next, as usual, given a module V over C , one defines $V^{\otimes(1)} = V$, and inductively

$$V^{\otimes(k)} = V \otimes V^{\otimes(k-1)}.$$

From what we just described, if V has a negation map $(-)$ then $V^{\otimes(k)}$ also has a natural negation map, and often is uniquely negated.

Now define the **negated tensor semialgebra** $T(V) = \bigoplus_n V^{\otimes(n)}$ (adjoining a copy of C if we want to have a unit element), with the usual multiplication. If V has a negation map then so does $T(V)$, induced from the negation maps on $V^{\otimes(k)}$; writing $\tilde{a}_k = a_{k,1} \otimes \cdots \otimes a_{k,k}$, we put

$$(-)(\tilde{a}_k) = (-)(a_{k,1} \otimes \cdots \otimes a_{k,k}).$$

9. LINEAR ALGEBRA OVER A UNIQUELY NEGATED TRIPLE

We can tackle the various notions of linear algebra over a system. Only the foundation is presented here; some deeper theorems and their subtleties involved are given in [5].

9.1. Matrices over systems.

Matrices over a uniquely negated system yield a system, as a special case of Example 5.9. Indeed, for $m = n^2$, as with classical algebra, $M_n(\mathcal{A}) \approx \text{End}_{\mathcal{A}}(\mathcal{A}^{(n^2)})$ has the module structure of $\mathcal{A}^{(n)}$ for any semiring \mathcal{A} , and we get a system, taking $\mathcal{T}(M_n(\mathcal{A})) = M_n(\mathcal{T})$, and defining $(-)$ and \preceq componentwise. Note that $(M_n(\mathcal{T}), \cdot)$ is no longer a monoid even when (\mathcal{T}, \cdot) is a monoid. Nevertheless, significant results are available, coupling standard computations with the transfer principle.

9.1.1. Determinants over an algebra with a negation map.

Definition 9.1. Suppose \mathcal{A} has a negation map $(-)$. For a permutation π , write

$$(-)^\pi a = \begin{cases} a & : \pi \text{ even;} \\ (-)a & : \pi \text{ odd.} \end{cases}$$

The $(-)$ -**determinant** $|A|$ of a matrix A is

$$\sum_{\pi \in S_n} (-)^\pi \left(\prod_i a_{i, \pi(i)} \right).$$

The **even part** is $\sum_{\pi \in S_n \text{ even}} \left(\prod_i a_{i, \pi(i)} \right)$, and the **odd part** is $\sum_{\pi \in S_n \text{ odd}} \left(\prod_i a_{i, \pi(i)} \right)$.

A matrix A is **nonsingular** if $|A| \in \mathcal{T}$. A is **singular** if $|A| \notin \mathcal{T}$.

When the system has height 2, the matrix A is singular iff $|A| \in \mathcal{A}^\circ$, which “explains” why this variant of the definition appears both in the supertropical and symmetrized literature.

We extend \preceq_\circ componentwise.

Lemma 9.2. If $(a_{i,j}) \preceq (b_{i,j})$ then $|(a_{i,j})| \preceq |(b_{i,j})|$.

Proof. Match the sums and products in the formula. □

Lemma 9.3. The $(-)$ -determinant is linear in any given row or column.

Proof. Same as for the classical situation. □

Lemma 9.4. If two rows or columns of a matrix A are the same, then A is singular.

Proof. The formula for the $(-)$ -determinant partitions into pairs of opposite $(-)$. □

Proposition 9.5. If the first row v_1 of A \circ -surpasses a linear combination of the other rows v_2, \dots, v_n , then A is singular.

Proof. Breaking up the first row, we see that A is a sum of matrices in which either the first row is in $\mathcal{A}^{(n)\circ}$ or is a scalar multiple of another row, so $|A|$ is a sum of elements of \mathcal{A}° . □

Definition 9.6. Write $a'_{i,j}$ for the $(-)$ -determinant of the j, i minor of A . The $(-)$ -**adjoint** matrix $\text{adj}(A)$ is $(a'_{i,j})$.

Remark 9.7. $|a_{i,j}| = \sum_{j=1}^n (-)^{i+j} a'_{i,j} a_{i,j}$, for any given i .

9.1.2. Invertible matrices.

Dolzán and Oblak [21] develop the tie between anti-negated triples (Definition 6.44) with matrix theory, by showing that the only invertible matrices over multiplicatively cancellative anti-negated semirings are generalized permutation matrices, a key feature in tropical algebra. Let us formulate this in terms of triples.

Proposition 9.8. *Over an anti-negated meta-tangible triple with $\mathbf{n} \neq \mathbf{0}$ for each $n \in \mathbb{N}$, the only invertible matrices are the tangible matrices.*

Proof. In view of Lemma 6.45, the proof in [73] goes through. \square

9.1.3. Some \preceq_{\circ} -identical relations for matrices.

Identical relations of $n \times n$ matrices can be translated (matching the matrix entries) into n^2 identities in commuting indeterminates. Using the transfer principle, we see that many identities of matrices over rings translate to \preceq_{\circ} -identical relations of $\mathbb{N}_{\max}[\Lambda]$. In accordance with Remark 7.20 that all of the coefficients in the classical formula for determinant are ± 1 , [2, Corollary 4.18] shows that $|A| \preceq_{\circ} A \operatorname{adj}(A)$ and $|A||A'| \preceq_{\circ} |AA'|$. This follows from the computations in [65] (also [34]), and we observe now that the same proofs hold more generally over \mathcal{T} -semirings since, although they are formulated there for semirings, they do not rely on distributivity, and only use the axioms set forth here over \mathcal{T} -semirings.

Proposition 9.9. $\operatorname{adj}(B) \operatorname{adj}(A) \preceq \operatorname{adj}(AB)$.

Proof. Writing $AB = (c_{i,j})$, we see that $\operatorname{adj}(AB) = (c'_{j,i})$ whereas the (i,j) -entry of $\operatorname{adj}(B) \operatorname{adj}(A)$ is $\sum_{k=1}^n b'_{k,i} a'_{j,k}$. Since $a'_{j,k} b'_{k,i}$ appears in $c'_{j,i}$, we need only check that the other terms in $c'_{j,i}$ occur in matching pairs with opposite signs. These are sums of products the form

$$d_{k_1, \pi(k_1)} d_{k_2, \pi(k_2)} \cdots d_{k_{n-1}, \pi(k_{n-1})},$$

where $k_t \neq j$, $\pi(k_t) \neq i$ for all $1 \leq t \leq n-1$, and

$$d_{k_t, \pi(k_t)} = a_{k_t, \ell} b_{\ell, \pi(k_t)}.$$

If the ℓ do not repeat, we have a term from $\operatorname{adj}(B) \operatorname{adj}(A)$. But if some ℓ repeats, i.e., if we have

$$d_{k_t, \pi(k_t)} = a_{k_t, \ell} b_{\ell, \pi(k_t)}, \quad d_{k_u, \pi(k_u)} = a_{k_u, \ell} b_{\ell, \pi(k_u)},$$

then in computing $c'_{j,i}$ we also have a contribution from σ where $\sigma(k_t) = \pi(k_u)$ and $\sigma(k_u) = \pi(k_t)$ (and $\sigma = \pi$) on all other indices, whereby we get

$$a_{k_t, \ell} b_{\ell, \sigma(k_t)} a_{k_u, \ell} b_{\ell, \sigma(k_u)} = a_{k_t, \ell} b_{\ell, \pi(k_u)} a_{k_u, \ell} b_{\ell, \pi(k_t)} = a_{k_t, \ell} b_{\ell, \pi(k_t)} a_{k_u, \ell} b_{\ell, \pi(k_u)},$$

as desired. \square

Lemma 9.10. $|A|I \preceq A \operatorname{adj}(A)$ over any triple.

Proof. The diagonal terms are equal, by definition, and the extra terms off the diagonal are known to match, by rewording [65, Lemma 2]. \square

Theorem 9.11. $|A||B| \preceq |AB|$, for any matrices $A, B \in M_n(\mathcal{A})$.

Proof. We appeal to the semiring argument taken from [65], matching terms in the products, since any term in $\det(AB)$ not in $\det(A)\det(B)$ occurs twice, with opposing signs. This follows from [65, p.352, end of proof of (a)]. \square

9.1.4. Prominent matrix monoids.

In order for this theory to be at our disposal for $M_n(\mathcal{T})$ in general, we pass to $M_n(\widehat{\mathcal{T}})$, with the switch negation map, in which case the $(-)$ -determinant is as in [2]. Namely, we define $|A|_{\circ} = |(A, (\mathbf{0}))|$. This is an ordered pair (a_0, a_1) , where a_0 is the even part of the determinant and a_1 is the odd part. These considerations lead us to define:

Definition 9.12. $\operatorname{SLS}_n(\hat{\mathcal{A}}) = \{A \in M_n(\mathcal{A}) : |A|_{\circ} \succeq_{\circ} (\mathbf{1}, \mathbf{0})\}$.

This is essentially the definition used in [46]. It contains all the elementary matrices, but is not generated by them, cf. [61]. Just as $\mathrm{SL}_n(\mathcal{A})$ is a classical algebraic group, with its symmetrized version given in Definition 9.12, we can define $\mathrm{PSL}_n(\mathcal{A})$ by taking $\mathrm{SLS}_n(\mathcal{A})$ modulo the congruence $\{(A, \alpha A) : \alpha \in \mathcal{A}\}$. We get versions of the other algebraic groups, by utilizing involutions.

Using Example 5.21 we can define \preceq -orthogonal matrices via the condition $(I, (0)) \preceq AA^t, A^t A$, and thereby define the \preceq -orthogonal semigroups, special \preceq -orthogonal semigroups, and analogously \preceq -symplectic semigroups.

9.2. Dependence relations of vectors.

A vector $v \in M$ is called **tangible** if each of its entries is in \mathcal{T}_0 . Thus, a matrix is tangible iff each of its rows is tangible.

Definition 9.13. *Suppose M is an \mathcal{A} -module.*

A set $S \subseteq M$ is \mathcal{T} -dependent if there are $v_1, \dots, v_m \in S$ and (nonzero) $\alpha_j \in \mathcal{T}$ such that

$$\sum_{j=1}^m \alpha_j v_j \in M^\circ.$$

Otherwise S is \mathcal{T} -independent.

An element $a \in M$ is \mathcal{T} -dependent on a \mathcal{T} -independent set $S \subseteq M$, written $a \in_{\mathrm{dep}} S$, if $S \cup \{a\}$ is \mathcal{T} -dependent.

*An element $a \in M$ is **strongly \mathcal{T} -dependent** on a \mathcal{T} -independent set $S \subseteq M$, written $a \in_{\mathrm{dep}} S$, if there are $v_1, \dots, v_m \in S$ and (nonzero) $\alpha_j \in \mathcal{T}$ such that*

$$a \preceq \sum_{j=1}^m \alpha_j v_j.$$

\mathcal{T} -dependence in a matroidal system is strong, and has many of the nice properties of a strong abstract dependence relation [66, Definition 6.2]. Note that when reversibility holds, we have $a \preceq b + \sum_i a_i$ if and only if $b \preceq a(-) \sum_i a_i$.

Proposition 9.14. *Suppose $(\mathcal{A}, \mathcal{T}, (-), \preceq)$ is a matroidal system. Then \mathcal{T} -dependence satisfies the Steinitz exchange property ([66, Definition 6.2, (AD2)]): If $a \in_{\mathrm{dep}} \{s\} \cup S$ and $a \not\preceq S$, with $(-)S = S$, then $s \in_{\mathrm{dep}} \{a\} \cup S$.*

Proof. If $\alpha a \preceq \beta s + \sum_{j=1}^m \alpha_j v_j$ then

$$\beta s \preceq \alpha a(-) \sum_{j=1}^m \alpha_j v_j = \alpha a + \sum_{j=1}^m \alpha_j ((-)s_j).$$

□

Let us consider the other conditions of [66, Definition 6.2]. If $v \in S$ then $v = v$ implies $v \in_{\mathrm{dep}} S$, which is (AD1). The finiteness condition (AD4) is by definition. But ironically (AD3), transitivity, may fail. If $a \in_{\mathrm{dep}} S$ and $S \in_{\mathrm{dep}} T$, then writing

$$\alpha a \preceq \sum_{j=1}^m \alpha_j v_j$$

for $v_j \in S$ and $\beta_j v_j \preceq \sum_{k=1}^{m_j} \gamma_{j,k} t_{j,k}$ for $t_{j,k} \in T$, each $1 \leq j \leq m$, we have

$$\alpha \beta_1 \cdots \beta_m a \preceq \sum_{j=1}^m \sum_{k=1}^{m_j} \beta_1 \cdots \beta_{j-1} \beta_{j+1} \cdots \beta_m \gamma_{j,k} t_{j,k}.$$

One would like to conclude that $a \in_{\mathrm{dep}} T$. The difficulty with such an argument is that some of the $t_{j,k}$ might repeat, so combining coefficients might take the sum out of \mathcal{T} . One could obtain a more technical version of transitivity by restricting the coefficients appearing in the individual dependence relations, but at the end is stuck with the counterexample of [41, Example 4.8]. In fact, \mathcal{T} -dependence is closely linked to tropical dependence from [47, Definition 6.3], which also is not necessarily transitive.

9.3. Ranks of matrices.

Our next task is to compare different notions of rank of matrices, in terms of its row vectors and its column vectors. We only consider tangible matrices A , i.e., with entries in \mathcal{T}_0 , for meta-tangible systems. This is a small step back from [48], but the tangible case is the compelling one, since one can recover the full supertropical theory from it.

Definition 9.15. *The (surpassing) row rank of a matrix A is the maximal number of \mathcal{T} -independent rows of A . The column rank of the matrix A is the maximal number of \mathcal{T} -independent columns of A .*

The submatrix rank of the matrix A is the maximal k such that A has a nonsingular $k \times k$ submatrix.

Let us consider the following assertions:

- (i) **Condition A1:** The submatrix rank is less than or equal to the row rank and the column rank.
- (ii) **Condition A2:** The three definitions of rank are equal for any tangible matrix, when \mathcal{T} is a multiplicative group.

9.3.1. Results for Condition A1.

A reasonably easy induction argument enables one to reduce Condition A1 to proving that a square matrix A is singular if its rows are dependent, which is our next result.

Theorem 9.16. *If the rows of a tangible $n \times n$ matrix A over a meta-tangible system are dependent, then $|A| \in \mathcal{A}^\circ$.*

Proof. Normalizing, we may assume that the sum of the rows are in $(\mathcal{A}^{(n)})^\circ$. Assume on the contrary that $|A| \in \mathcal{T}$, and take k_1, \dots, k_n such that $|A| = a_{k_1,1} \cdots a_{k_n,n}$. In each column we take a minimal nontangible sum of tangible elements including $a_{k_j,j}$. Namely, inductively, take $i_{j,1} = k_j$, and given $I_{j,m-1} = \{i_{j,1}, \dots, i_{j,m-1}\}$, we put

$$I_{j,m} = I_{j,m-1} \cup \{i_{j,m}\} = \{i_{j,1}, \dots, i_{j,m}\},$$

where $i_{j,m}$ is chosen such that

$$\begin{cases} \sum_{u=1}^m a_{i_{j_u},j} \in \mathcal{A}^\circ & \text{if such } a_{i_{j,m},j} \text{ exists,} \\ \text{otherwise } i_{j,m} \notin I_{j,m-1} \text{ is arbitrary (and we define } I_j := I_{j,m} \text{ and terminate the process).} \end{cases}$$

By Proposition 6.5, $a_{i_{j_m},j} = (-)^{\sum_{u=1}^{m-1} a_{i_{j_u},j}}$. Let A' denote the matrix for which we replace $a_{i,j}$ by 0 for each $i \notin I_j$.

Since $a_{k_1,1} \cdots a_{k_n,n}$ already accounts for $|A|$, we have $|A'| = a_{k_1,1} \cdots a_{k_n,n} = |A|$. But viewing the entries as indeterminates, we know that the classical determinant of A' is 0 , so, by the transfer principle, $|A'| \in \mathcal{A}^\circ$. \square

9.3.2. Partial results for Condition A2.

Proposition 9.17. *Condition A2 holds for any tangible 2×2 matrix.*

Proof. Let $v_i = (a_{i,1}, a_{i,2})$ be the rows of the matrix $A = (a_{i,j})$ for $i = 1, 2$. If $a_{1,1} = 0$, then $|A| = (-)a_{1,2}a_{2,1}$ which cannot be in \mathcal{A}° unless $a_{1,2}$ or $a_{2,1}$ is 0 , in which case the proposition is clear. Hence we may assume that $a_{1,1} \neq 0$, and by symmetry we may assume that each $a_{i,j} \neq 0$. Now we may normalize (multiplying row i by $a_{i,1}^{-1}$) and assume that $a_{i,1} = 1$ for $i = 1, 2$. But then $|A| = a_{1,2}(-)a_{2,2}$, which is in \mathcal{A}° iff $a_{1,2} = a_{2,2}$. \square

This proof would seem to contain the kernel of a proof of Condition A2 for general n , but Gaubert pointed out that a counterexample to Condition A2 already appears in [3]; this is discussed and generalized in [5], continuing the construction of §1.4.3. Here is a tantalizing observation.

Proposition 9.18. *If $|A| \in \mathcal{A}^\circ$ and $v_i = (a_{i,1}, \dots, a_{i,n})$ are the rows of A , then $\sum_{j=1}^n (-)^{i+j} a'_{1,j} v_i \in \mathcal{A}^\circ$, where $a'_{i,j}$ is as in Definition 9.6.*

Proof. The first column of $\sum_{j=1}^n (-)^{i+j} a'_{i,j} v_i$ is $|A|$, given in \mathcal{A}° , and the other columns are in \mathcal{A}° because $\sum_{j=1}^n (-)^{i+j} a'_{1,j} v_i$ is the determinant of the matrix obtained by replacing the j column of A by its first column, so by Lemma 9.4 this combination is in \mathcal{A}° . \square

Unfortunately, we have no assurance that the $a'_{1,j} \in \mathcal{T}$ when $n > 2$, which fails in the counterexamples, but this observation is used in verifying Condition A2 for $(-)$ -bipotent systems of the first kind, cf. [5].

10. TROPICALIZATION

Tropicalization, perhaps the main tool in tropical mathematics, has been studied in various contexts. Originally “standard” tropicalization went to the max-plus algebra, by applying logarithms to varieties defined over from \mathbb{R} , as exposed in [37]. Most recent research has focused on the Puiseux series valuation to be recalled presently, in the following variants:

Example 10.1.

- (i) *The Puiseux series valuation to the max-plus algebra from the Puiseux series algebra $K\{\{t\}\}$ on the variable t , again as exposed in [37], and to be reviewed presently.*
- (ii) *The Puiseux series valuation to the supertropical algebra, [47].*
- (iii) *The Puiseux series valuation to the “exploded algebra” from the Puiseux series algebra $K\{\{t\}\}$ on the variable t , cf. [62].*
- (iv) *The Puiseux series valuation to the algebra layered by \mathbb{N} , [40].*
- (v) *The Puiseux series valuation to the symmetrized algebra, [3].*

Each version has its specific motivation. Supertropical algebra is compatible with the value group of the Puiseux series valuation. If one wants to take the residue field into account one would pass to the exploded (layered) algebra. Even so, this only takes the lowest term of the Puiseux series into account. When this is lost, one would need to dig deeper into the Puiseux series, taking an infinite direct sum $\oplus_{i \in \mathbb{N}} \mathcal{A}_i$ of layered systems, passing from one component to another as one enters \mathcal{A}_i° . This would be the tropicalization of the associated valuation ring, but so far it has not appeared in the literature.

Layered systems are geared for derivatives and other aspects of differential geometry at the tropical level.

On the other hand, the symmetrized system seems best for handling tropicalization of determinants, since it enables one to treat both the positive and negative parts in the formula.

In this section we unify these various approaches by casting tropicalization in terms of morphisms of systems. But since tropicalization relies so heavily on the Puiseux series valuation, we pause for some observations about valuations.

10.1. Tropicalization of Puiseux series.

For any semiring[†] K of a given signature, one can define the set $K\{\{t\}\}$ of Puiseux series on the variable t , which is the set of formal series of the form $f = \sum_{k=\ell}^{\infty} c_k t^{k/N}$ where $N \in \mathbb{N}$, $\ell \in \mathbb{Z}$, and $c_k \in K$. (One could use any subgroup of $(\mathbb{R}, +)$ for the exponents in the series, but the definition becomes more complicated without enhancing the theory since $(\mathbb{Q}, +)$ is model complete in the elementary theory of ordered groups.) Then we have the **Puiseux valuation** $\text{val} : K\{\{t\}\} \setminus \{0\} \rightarrow \mathbb{Q}_{\max} \subset \mathbb{R}_{\max}$ defined by

$$\text{val}(f) = \min_{c_k \neq 0} \{k/N\}, \quad (10.1)$$

and formally $\text{val}(0) = 0 (= -\infty)$. We also call val **tropicalization**.

Customarily one takes K to be the field of complex numbers, so that $K\{\{t\}\}$ is an algebraically closed field, but we find it convenient to consider tropicalization over any semiring[†], especially \mathbb{N}_0 . We would want val to be a morphism. But this does not quite work since \mathbb{Q}_{\max} does not have negatives, so we consider several related versions of tropicalization which are more amenable to algebraic methods.

Proposition 10.2. *In each of the following cases taken from Example 10.1 (in the same order), v provides a morphism v from the Puiseux series $K\{\{t\}\}$ (viewed as a classical system) to \mathcal{T} in one of the meta-tangible systems we have described in the previous sub-sections:*

- (i) $v(f) = -\text{val}(f)$, taking $(\mathcal{A}, \mathcal{T}, (-), \preceq)$ to be the max-plus algebra, cf. Remark 1.23.
- (ii) $v(f) = -\text{val}(f)$, taking $(\mathcal{A}, \mathcal{T}, (-), \preceq)$ to be the supertropical algebra.
- (iii) Given a Puiseux series $f = \sum_{k=\ell}^{\infty} c_k t^{k/N}$, $v(f) = (c_{\text{val}(f)}, -\text{val}(f))$, in \mathcal{T} of the ELT algebra \mathcal{A} , where ℓ is the coefficient of the lowest (nonzero) term in the Puiseux series.

*This can be viewed more generally, in analogy to viewing tropicalization as passing to the target of a valuation $v : R \rightarrow \mathbb{Q}$, where R is a ring. Suppose that the valuation v has a **uniformizer** π such that $v(\pi) = 1$. Thus, for any element r , taking $v(r) = m/n$ we have $v(\pi^{-m/n} r) = 0$. Now*

- one can also take into account the residue ring R/P where P is the valuation ideal and, letting $L = R/P$, consider the map $R \rightarrow L \times \mathcal{G}$, the *ELT-algebra*, given by $a \mapsto (\pi^{-m/n}r, v(r))$.
- (iv) $v(f) = (1, -\text{val}(f))$, taking $(\mathcal{A}, \mathcal{T}, (-), \preceq)$ to be the layered algebra.
 - (v) For K a subring of \mathbb{R} , \mathcal{A} as in Remark 7.18, where and $f = \sum_{k=\ell}^{\infty} c_k t^{k/N}$,

$$v(f) = \begin{cases} (-\text{val}(f), 0) & \text{for } c_{\text{val}(f)} > 0, \\ (0, -\text{val}(f)) & \text{for } c_{\text{val}(f)} < 0, \\ (0, 0) & \text{for } c_{\text{val}(f)} = 0. \end{cases}$$

(This raises the question of treating non-real fields, which is being considered by the *M-Plus* group in Paris.)

Proof. In each case, we verify that $v(-f) = (-)v(f)$, and v preserves addition (with respect to the relation \preceq). \square

10.2. Tropicalization of classical systems defined over Puiseux series.

The process of § 10.1 indicates a way of tropicalizing standard definitions of algebra in this setting, where one expects that some case of Proposition 10.2 is being used, according to context.

Remark 8.20(ii) provides the key. Suppose we have some signature of classical rings R (not necessarily associative). A fortiori, these are semialgebras defined over \mathbb{N}_0 , so can be thought of as systems. But the Puiseux series $R\{\{t\}\}$ (taking powers in some subgroup $(\mathcal{G}, +)$ of $(\mathbb{R}, +)$) is a semialgebra over \mathbb{N} , so we can take $R \otimes_{\mathbb{N}} K\{\{t\}\}$, which we call the Puiseux series over R , denoted as $R\{\{t\}\}$.

We define the tropical analog to be $\bar{R} \otimes_{\mathbb{N}} \mathcal{A}$, where \bar{R} is a suitable semiring[†] with negation map, defined as the analogous system to R , and \mathcal{A} is as in Proposition 10.2.

Proposition 10.3. *In the notation of Proposition 10.2, there is a natural \mathcal{T} -morphism $R\{\{t\}\} \rightarrow \bar{R} \otimes_{\mathbb{N}} \mathcal{A}$, sending $r \otimes f \rightarrow \bar{r} \otimes v(f)$.*

Proof. Taking the tensor product of the natural map $\mathbb{N} \rightarrow \mathbb{N}_{\max}$ with one of the tropicalization maps of Proposition 10.2 gives us the desired \mathcal{T} -morphism of systems from the classical version to the tropical version, in view of Remark 8.20. \square

In summary, we define some classical algebraic signature, re-express it as a system (e.g., a semialgebra with a negation map and a surpassing relation), and tensor it upstairs with Puiseux series and downstairs with the tropicalization of the Puiseux series. This process, which we call **universal tropicalization**, is the model for many of the subsequent examples in this section.

Remark 10.4. *Under this tropicalization map, we need not have $0 \mapsto 0$, since the definition of \mathcal{T} -morphism only requires the image of 0 to be a quasi-zero.*

11. TROPICAL STRUCTURES ARISING FROM TROPICALIZATION

Let us apply universal tropicalization of the previous section to obtain tropical analogs of classical algebraic structures.

11.1. Exterior (Grassmann) semialgebras with a negation map.

As in the classical case, for free modules, the tensor semialgebra yields a construction of the Grassmann semialgebra whose base is the union of even elements and odd elements.

The definition given in [28] (which goes on to treat the Plücker equations) is a semialgebra generated by a free module V with a base $\{e_i : i \in I\}$, together with a product $\mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ satisfying $e_i^2 = 0$ for each $i \in I$.

These could be constructed by means of the tensor semialgebra, modulo the relations $x_i^2 = 0$. As noted in [28], the definition given there relies heavily on the presentation in terms of the base, since in general $v^2 \neq 0$ for $v \in V$. This would mean that a sub-semialgebra of a Grassmann algebra need not be Grassmann. In other words, the semialgebra generated by e_1 and $e_1 + e_2$ is not Grassmann. This can be rectified, as indicated by the use of Proposition 10.3, and keeping Remark 10.4 in mind.

Definition 11.1. *A (faithful) **Grassmann**, or **exterior**, semialgebra, over a triple $(V, \mathcal{T}, (-))$, where V is a module, is a semialgebra \mathcal{A} generated by V , together with a negation map extending $(-)$ and a product $\mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ satisfying*

$$(i) \quad v^2 \in \mathcal{A}^\circ \quad \text{for} \quad v \in V; \quad (11.1)$$

$$(ii) \quad (-)(v_1 \cdots v_t) = ((-)v_1)v_2 \cdots v_t; \quad (11.2)$$

$$(iii) \quad v_1 v_2 = (-)v_2 v_1 \quad \text{for} \quad v_i \in V. \quad (11.3)$$

Thus $v_{\pi(1)} \cdots v_{\pi(t)} = (-)^t v_1 \cdots v_t$.

When V is the free module, this definition covers the one in [28], in which $(-)$ is the identity map, and $e_i(-)e_i$ is sent to 0. Thus Definition 11.1 maps onto [28]. Their techniques can be adapted to this situation.

The appropriate triple is $(\mathcal{A}, V, (-))$, where $\mathcal{T}(\mathcal{A}) = \{v_1 \cdots v_t : v_i \in \mathcal{T}\}$.

Lemma 11.2. *If V is the free module with negation, with base $\{e_i, (-)e_i : i \in I\}$, then it is enough to check that*

$$e_i^2 \in \mathcal{A}^\circ, \quad e_i e_j = (-)e_j e_i, \quad \forall i, j \in I.$$

Proof. $e_i e_j + e_j e_i = e_i e_j (-)e_i e_j \in \mathcal{A}^\circ$, so $(\sum \alpha_i e_i)^2 = \sum \alpha_i^2 e_i^2 + \sum_{i < j} \alpha_i \alpha_j (e_i e_j + e_j e_i) \in \mathcal{A}^\circ$, yielding (i). For (ii), we note that $e_i^2 = (-)e_i^2$ since $e_i^2 \in \mathcal{A}^\circ$. \square

Lemma 11.3. $v_1 v_2 = (-)v_2 v_1$ is central in \mathcal{A} , for all $v_1, v_2 \in V$.

Proof. $v_1 v_2 v_3 = (-)v_1 v_3 v_2 = v_3 v_1 v_2$, implying that $v_1 v_2$ is central. \square

Definition 11.4. *Given a Grassmann semialgebra G over a module V with a negation map $(-)$, we define G_0 to be the submodule generated by all even products of elements of V , and G_1 to be the submodule generated by all odd products of elements of V .*

Lemma 11.5. $G = G_0 + G_1$. G_0 is in the center of G , and $G_1 = G_0 V$. When V is the free module with negation, then $G = G_0 \oplus G_1$.

Proof. The first assertion is an immediate induction based on Lemma 11.3. For the free module with negation, we match components. \square

Lemma 11.6. *If V is the free F -module with negation of Example 7.8, with base $\{e_i, (-)e_i : i \in I\}$, then any nonzero element is a sum of terms $(\pm)\alpha_{i_1} \cdots e_{i_k} + a^\circ$, where $i_1 < \cdots < i_k$, $\alpha \in F$, and $a \in \mathcal{A}$.*

Proof. By linearity, we may assume that $v_j = e_{i_j}$ for $j = 0, 1$. Rearrange the e_i appearing in the summands, since any time an e_i repeats, the product is in \mathcal{A}° . \square

Example 11.7. *Suppose F is a semifield[†]. When V is the free F -module with negation, with base $\{e_i, (-)e_i : i \in I\}$, the tensor semialgebra $T(V)$ becomes a Grassmann semialgebra \mathcal{A} when we impose the extra relations that $e_j e_i = ((-)e_i)e_j = e_i((-)e_j)$ for all $i, j \in I$. $\mathcal{T}(T(V))$ is the set of simple tensors in which one does not have both e_i and $(-)e_i$. Every term of even degree in the e_i is central, so \mathcal{A} satisfies the \succeq_\circ -surpassing identical relation $[x_1, [x_2, x_3]] \succeq 0$.*

The set $\mathcal{T}(\mathcal{A})$ of tangible elements is the monoid generated by the tangible elements of V . Again, the surpassing relation extends naturally to \mathcal{A} , as a free module.

This inspires us to take an idea from [20] to get the “free” Grassmann construction. For convenience we take the default situation.

Definition 11.8. *The **extended Grassmann** semialgebra over an F -module V with a negation map $(-)$, is the free semialgebra \mathcal{A} generated by the $e_i, (-)e_i$ of Example 7.8 and central commuting indeterminates $\lambda_{j,k}$ (formally commuting with all $e_i, (-)e_i$), where we declare that $(-)\lambda_{j,i} = \lambda_{i,j}$, satisfying $e_i e_j = \lambda_{i,j}$ for all i, j .*

(This creates new identical relations such as $e_i \lambda_{j,k} = \lambda_{i,j} e_k$ for all i, j, k .)

Lemma 11.9. *The extended Grassmann semialgebra is isomorphic to the semialgebra we have defined in Example 11.7, where we identify $\lambda_{i,j}$ with $e_i e_j$. It also has an involution $(*)$ given by*

$$\left(\sum \alpha_i e_i (-) \alpha'_i e_i \right)^* = \sum \alpha'_i e_i (-) \alpha_i e_i.$$

Proof. $\lambda_{i,j}$ are central by construction, and

$$\lambda_{j,i} = e_j e_i = (-) e_i e_j = (-) \lambda_{i,j}.$$

The last assertion is clear. \square

11.2. Nonassociative algebras with a negation map.

We next bring in Lie semialgebras via tropicalization, since Lie algebras are so important in classical representation theory. Whereas the Jacobi identity on a Lie algebra L is equivalent to the adjoint representation $L \rightarrow \text{ad } L$ being a Lie homomorphism, the correspondence in tropical algebra is more delicate, and we pause for a general discussion of nonassociative semialgebras and their adjoint algebras.

Definition 11.10. A semialgebra \mathcal{A} with negation map is **anticommutative** if it satisfies the conditions for all $a, b \in \mathcal{A}$:

- (i) $a^2 \in \mathcal{A}^\circ$;
- (ii) $ba = (-)(ab) = a((-)b) = ((-)a)b$.

(In classical mathematics, (ii) is derived from (i) by multilinearization, but this argument requires a genuine negative, and so is inapplicable here.)

Given any (nonassociative) semialgebra \mathcal{A} over a semifield[†] F and $a \in \mathcal{A}$, we define $\text{ad}_a \in \text{End}_F \mathcal{A}$ by $\text{ad}_a(b) = ab$, and

$$\text{ad}\mathcal{A} = \{f \in \text{End}_F \mathcal{A} : f \succeq_\circ \text{ad}_a \text{ for some } a \in \mathcal{A}\}.$$

$\text{ad}\mathcal{A}$ is a submodule of $\text{End}_F \mathcal{A}$, with a negation map, even though $\text{ad}\mathcal{A}$ need not be a semialgebra.

Lemma 11.11. $(-)\text{ad}_a = \text{ad}_{(-)a}$.

Proof. $(-)\text{ad}_a(b) = (-)ab = \text{ad}_{(-)a}(b)$. \square

11.2.1. Super-semialgebras.

As in the classical case, one can “superize” the various classes in universal algebra, mimicking the standard classical way of making a theory super.

Definition 11.12. The **Grassmann envelope** of a super-semialgebra $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ is the subalgebra $(\mathcal{A}_0 \otimes G_0) \oplus (\mathcal{A}_1 \otimes G_1)$ of $\mathcal{A} \otimes G$, with G as in Lemma 11.5. (Thus we view the Grassmann envelope without the grading.)

Suppose \mathcal{V} is a variety of universal algebras. A **super- \mathcal{V}** algebra is a super-semialgebra \mathcal{A} whose Grassmann envelope is in \mathcal{V} .

For example, the Grassmann envelope of G itself is $(G_0 \otimes G_0) \oplus (G_1 \otimes G_1)$ which is commutative, justifying our earlier use of the term “super-commutative.” Conceptually, Definition 11.12 is just an elegant form of book-keeping, where in evaluating multilinear operations on a superalgebra we put in $(-)^k$, where k is the number of odd occurrences of the entries.

11.3. Lie semialgebras and Lie super-semialgebras.

We turn again to Proposition 10.3 for the tropical version of Lie algebras.

Definition 11.13. A **Lie semialgebra with a negation map (over a semiring \mathcal{A})** is a module L with a negation map $(-)$, endowed with anticommutative multiplication $L \times L \rightarrow L$, written as $(a, b) \mapsto [ab]$, called a **Lie bracket** (in view of the standard notation $[ab]$ for Lie multiplication), satisfying $\text{ad}_{[ab]} \preceq [\text{ad}_a, \text{ad}_b]$ for all $a, b \in L$. (Note that we do not require a negation map on \mathcal{A} .)

Lemma 11.14. ad is a morphism from L to $\text{End}_F \mathcal{A}$. (In fact ad preserves addition.) $[[ab]v] \preceq [a[bv]](-)[b[av]]$ for all $a, b, v \in L$.

Proof. Follows from the definitions. \square

This can be viewed as the \preceq_\circ -surpassing version of Jacobi’s identity.

This is a bit stronger than the analog of Blachar’s definition [11], but is satisfied by the following key example:

Proposition 11.15. Any associative semiring[†] R with negation map becomes a Lie semialgebra under the Lie product $[ab] = [a, b]$.

Proof. Follows at once from the strong transfer principle applied to the usual Jacobi identity for the special Lie algebra of an associative algebra; alternatively, one could use Lemma 1.42. \square

We call this Lie semialgebra R^- .

Corollary 11.16. *For any associative semiring[†] $(R, *)$ with involution and negation map, $(R, *)^-$ is a Lie sub-semialgebra of R^- . under the Lie product $[ab] = [a, b]$.*

Proof. It is closed under the Lie product. \square

Putting everything together and recalling Definition 8.6 yields

Proposition 11.17. *If L is a Lie semialgebra with a negation map, then $\text{ad } L$ is a Lie sub-semialgebra of $\text{End}_F L$, and there is a Lie \preceq -morphism $L \rightarrow \text{ad } L$, given by $a \mapsto \text{ad}_a$.*

Remark 11.18. *We are now in a position to define the symmetrized analogs of the classical Lie algebras over a semiring[†] \mathcal{A} . Namely, we take $\widehat{\text{sl}}_n(\mathcal{A}) = \{((a_{i,j}), (b_{i,j})) \in M_n(\hat{\mathcal{A}}) : \sum_i a_{i,i} = \sum_i b_{i,i}\}$, the symmetrized analog of the classical Lie algebra A_{n-1} . To obtain the analogs of B_n , C_n , and D_n , one just applies Corollary 11.16 to the transpose and symplectic involutions, taking the subset $\{(A, A^*) : A \in \mathcal{M}_n(\hat{\mathcal{A}})\}$:*

- *We get the symmetrized version of the classical Lie algebra B_n when $(*)$ is the transpose and n is odd.*
- *We get the symmetrized version of the classical Lie algebra C_n when $(*)$ is the symplectic involution and n is even.*
- *We get the symmetrized version of the classical Lie algebra D_n when $(*)$ is the transpose and n is even.*

11.3.1. Lie super-semialgebras.

Let us superize the Lie theory by means of Definition 11.12.

Definition 11.19. *A **Lie super-semialgebra with a negation map** is a module L with a negation map $(-)$, endowed with (super-anticommutative) multiplication $L \times L \rightarrow L$, written as $(a, b) \mapsto [ab]_s$, called a **superLie bracket**, satisfying $[[ab]_s v]_s \preceq [a[bv]_s]_s (-)[b[av]_s]_s$ for all homogeneous $a, b, v \in L$.*

Thus, for all $a, b, v \in L$ we have $[[ab]_s v]_s \preceq [a[bv]_s]_s (-)[b[av]_s]_s$. (The negations all appear in the same degree in the super-version, so cancel out.)

Proposition 11.20. *Any associative semiring[†] R with negation map becomes a Lie super-semialgebra under the super-Lie bracket*

$$[a_i a_j]_s = a_i a_j (-)^{ij} a_j a_i, \quad a_i \in R_i, \quad a_j \in R_j. \quad (11.4)$$

Proof. Reread the Leibniz identities (Lemma 1.42) in terms of (11.4). \square

11.4. Poisson algebras and their module congruences.

We turn again to Proposition 10.3.

Definition 11.21. *A **Poisson semialgebra** is an associative semialgebra \mathcal{A} with a negation map, together with a bilinear operation $\{ , \} : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$, called a **Poisson bracket**, satisfying*

$$\{ab, c\} \preceq a\{b, c\} + \{a, c\}b, \quad \{a, bc\} \preceq \{a, b\}c + b\{a, c\}, \quad \forall a, b, c \in \mathcal{A}.$$

(This takes into account Definition 11.13, as well as Proposition 11.15.) Then $\{ , \}$ yields a Lie structure as in Proposition 11.15.

Example 11.22. *All of the following are commutative Poisson algebras.*

- (i) *If L is a f.d. Lie algebra with negation map, having base a_1, \dots, a_n , then, viewing the a_i as commuting indeterminates in the commutative polynomial algebra $R = F[a_1, \dots, a_n]$, introduce a Poisson structure on R by defining $\{a_i, a_j\}$ to be the Lie product in L and extending the Poisson bracket via the Leibniz identities, i.e.,*

$$\{ab, c\} = a\{b, c\} + \{a, c\}b, \quad \{a, bc\} = \{a, b\}c + b\{a, c\}, \quad \forall a, b, c \in \mathcal{A}.$$

- (ii) Suppose V is a f.d. vector space with an alternating bilinear form. Take a base $\{x_1, \dots, x_n\}$ of V . The polynomial algebra $F[x_1, \dots, x_n]$ becomes a Poisson algebra, where one defines $\{x_i, x_j\}$ to be $\langle x_i, x_j \rangle$.
- (iii) If R is an associative semialgebra with negation map, filtered by \mathbb{N} , such that the associated graded algebra $\text{gr}(R)$ is commutative, then $\text{gr}(R)$ has a Poisson bracket defined as follows: For $a + R_{i-1} \in \text{gr}(R)_i$ and $b + R_{j-1} \in \text{gr}(R)_j$, define $\{a, b\}$ to be $[a, b] + R_{i+j-2} \in \text{gr}(R)_{i+j-1}$.

The super-version is obtained by taking instead Definition 11.19 and Proposition 11.20.

12. APPENDIX A: HYPERFIELDS AS SYSTEMS

As in the main text, viewing a hyperfield as a multiplicative group with additive structure on part of its power set, we want to extend this structure to all of the power set, thereby making the definitions tighter and “improving” the additive structure to make standard tools more available.

The main theme of this appendix is to examine precisely how the category of (canonical) hypergroups embeds into the category of (uniquely negated) systems, specifically \mathcal{T} -semirings with negation (and their modules), where \mathcal{T} is the hyperfield and $\mathcal{P}(\mathcal{T})$ is its power set, with special attention to the more prominent examples given in Examples 12.8.

As noted earlier, the tricky part is distributivity, which has several versions.

Definition 12.1. Define the following notions, where a pre-semiring \mathcal{A} acts on a semigroup $(S, +)$, for $a_i \in \mathcal{A}$ and $s_j \in S$:

- (i) **Left distributivity:** $(a_1 + a_2)s = a_1s + a_2s$.
- (ii) **Right distributivity:** $a(s_1 + s_2) = as_1 + as_2$.
- (iii) **Two-sided distributivity:** Left and right distributivity.
- (iv) **Double distributivity** $(a_1 + a_2)(s_1 + s_2) = a_1s_1 + a_2s_1 + a_1s_2 + a_2s_2$.
- (v) **Generalized distributivity:**

$$\left(\sum_i a_i \right) \left(\sum_j s_j \right) = \sum_{i,j} (a_i s_j).$$

The last four properties are in order of increasing formal strength (although by induction, double distributivity implies generalized distributivity). A semiring satisfies generalized distributivity. But hypergroups make us encounter structures for which there is a difference. We also define **weak distributivity** in a system via $(a_1 + a_2)s \preceq a_1s + a_2s$.

12.1. Power sets of semigroups.

To proceed further, we need associativity at the level of sets, and we need the following definition to make this precise (and hopefully more manageable, since then we can do all the calculations in the power set).

Definition 12.2. We call $\mathcal{P}(\mathcal{T})$ a **power semigroup** when $\mathcal{P}(\mathcal{T})$ is a semigroup with respect to a commutative associative binary operation $\mathcal{P}(\mathcal{A}) \times \mathcal{P}(\mathcal{A}) \rightarrow \mathcal{P}(\mathcal{A})$, compatible with the element operation, in the sense that

$$S_1 \boxplus S_2 = \bigcup \{ \{s_1\} \boxplus \{s_2\} : s_i \in S_i \} \quad (12.1)$$

($\{0\}$ is the neutral element.)

For the other operators, including multiplication, we require that

$$\omega_{m,j}(S_{1,j}, \dots, S_{m,j}) = \bigcup \{ \omega_{m,j}(s_{1,j}, \dots, s_{m,j}) : s_{k,j} \in S_{k,j}, 1 \leq k \leq m \}. \quad (12.2)$$

The connection from hyperrings to \mathcal{T} -semirings comes via the power set $\mathcal{P}(\mathcal{T})$. The sets $\{a\}$ for $a \in \mathcal{T}$ are called **singletons**. The hypermonoid of the power semigroup $\mathcal{P}(\mathcal{T})$ is the subset of $\mathcal{P}(\mathcal{T})$ containing all singletons. Let us lift familiar operations and their universal relations from \mathcal{T} to $\mathcal{P}(\mathcal{T})$.

Theorem 12.3. (i) Given a monoid $(\mathcal{T}, \cdot, \mathbb{1})$, we can extend the operation elementwise to $\mathcal{P}(\mathcal{T})$ by putting

$$S_1 S_2 = \{a_1 a_2 : a_j \in S_j\}$$

for $S_i \subseteq S$. Then $(\mathcal{P}(\mathcal{T}), +)$ also is a monoid, with the identity element $\{1\}$, on which \mathcal{T} acts.

Given a semigroup $(A, +)$, we can define addition elementwise on $\mathcal{P}(A)$ by defining

$$S_1 + S_2 = \{a_1 + a_2 : a_j \in S_j\}.$$

Then $(\mathcal{P}(A), +)$ also is a semigroup.

Thus, when A is a semiring, so is $\mathcal{P}(A)$.

- (ii) More generally, from the perspective of universal algebra, we can lift operators from an $(\Omega; \text{Id})$ -algebra A to $\mathcal{P}(A)$, as follows: Given an operator $\omega_{m,j} = \omega_{m,j}(x_{1,j}, \dots, x_{m,j})$ on A , we define $\omega_{m,j}$ on $\mathcal{P}(A)$ via

$$\omega_{m,j}(S_{1,j}, \dots, S_{m,j}) = \{\omega_{m,j}(a_{1,j}, \dots, a_{m,j}) : a_{k,j} \in S_{k,j}, 1 \leq k \leq m\}.$$

Then any multilinear universal relation holding in A also holds in $\mathcal{P}(A)$.

Proof. (i) We need to verify associativity and distributivity:

$$\begin{aligned} (S_1 + S_2) + S_3 &= \{a_1 + a_2 : a_j \in S_j\} + S_3 \\ &= \{(a_1 + a_2) + a_3 : a_j \in S_j\} \\ &= \{a_1 + (a_2 + a_3) : a_j \in S_j\} \\ &= S_1 + \{a_2 + a_3 : a_j \in S_j\} \\ &= S_1 + (S_2 + S_3). \end{aligned} \tag{12.3}$$

$$\sum S_i \sum T_j = \left\{ \sum_i a_i : a_i \in S_i \right\} \left\{ \sum_j b_j : b_j \in T_j \right\} = \left\{ \sum_{i,j} a_i b_j : a_i \in S_i, b_j \in T_j \right\} = \sum S_i T_j. \tag{12.4}$$

- (ii) We generalize the proof in (i). By an easy induction argument, any formula $\phi(x_1, \dots, x_\ell)$ satisfies

$$\phi(S_1, \dots, S_\ell) = \{\phi(a_1, \dots, a_\ell) : a_j \in S_j\},$$

and thus any multilinear universal relation $\phi = \psi$ holding elementwise in \mathcal{A} also holds set-wise in $\mathcal{P}(\mathcal{A})$. \square

In particular, distributivity lifts from a semiring A to $\mathcal{P}(A)$. But being a group does not lift, since the defining universal relation $xx^{-1} = 1$ is quadratic in x . (Here we are defining the inverse as a unary operation $\omega_1 : x \mapsto x^{-1}$, so the left side is $x\omega_1(x)$.) In fact the only invertible elements are the singletons.

If all of the products in A are singletons (which is our running assumption), then we have a monoid structure on A , and are back to Theorem 12.3.

12.1.1. Distributivity in the power set.

Remark 12.4. Viro showed that the general transition of universal relations to $\mathcal{P}(A)$ is not as straightforward as it may seem. The analogous argument to Theorem 12.3 unravels for hyperrings, since distributivity does not pass down to elements:

- (i) [74, Theorem 4.B] $(a \boxplus b)(c \boxplus d) \subseteq (ac) \boxplus (ad) \boxplus (bc) \boxplus (bd)$ in any hyperring;
- (ii) [74, Theorem 5.B] The “triangle” hyperfield R does not satisfy “double distributivity,” so $\mathcal{P}(R)$ is not distributive.

In other words, the analog of Theorem 12.3 fails for distributivity, which is why we introduced \mathcal{T} -semirings in the first place. To overcome this setback, we need to modify our underlying algebraic structure both at the hyper level and the power set level, the crux of the matter being distributivity. General distributivity is the property we need to weaken on sets $S = \{a_i : i \in I\}$ and $T = \{b_j, j \in J\}$. For any finite set $S = \{s_1, \dots, s_m\}$ we write $\boxplus S$ for $s_1 \boxplus \dots \boxplus s_m$, which makes sense since we have already proved associativity of \boxplus .

Actually, many of the important examples of hyperfields are doubly distributive, so we could pass to (distributive) power semirings without further ado. But even in the absence of doubly distributivity, we can formulate this in terms of universal algebra, in order to have those techniques at our disposal. On the face of it, this is problematic since the hypersum set could be arbitrarily large. However, we circumvent this difficulty by focusing on the monoid of singletons, and using operators instead of elements.

Definition 12.5. Suppose (\mathcal{T}, \cdot) is a monoid and (\mathcal{T}, \boxplus) a hypermonoid. Multiplication **weakly distributes** over \boxplus in \mathcal{T} if for all $a_i, b \in \mathcal{T}$ we have

$$(a_1 \boxplus a_2)b \subseteq (a_1b \boxplus a_2b). \quad (12.5)$$

In this case, when $(\mathcal{S}, \boxplus, \{0\})$ is closed under \boxplus and multiplication, we call \mathcal{S} a **weak power semiring**, or a **weak sub-semiring** of $\mathcal{P}(\mathcal{T})$.

Proposition 12.6. Suppose \mathcal{T} is a hyperring. Then, with the above operations, $(\mathcal{P}(\mathcal{T}), \cdot, \boxplus)$ is a \mathcal{T} -semiring.

Proof. We need to verify

$$(\boxplus S)(\boxplus T) \subseteq \boxplus(ST). \quad (12.6)$$

But writing $S = \{s_1, \dots, s_m\}$ we have

$$(\boxplus S)(\boxplus T) = \bigcup_i (\boxplus s_i)T = \bigcup_i \boxplus(s_iT) \subseteq \boxplus(ST)$$

since each $s_iT \subseteq \boxplus ST$. \square

The reverse inclusion fails since we simultaneously encounter s_iT for i varying. So far, we see that the power set of a hyperring is a weak power semiring. Now we repeat the proof of [74, Theorem 4.B], to show that at the bottom level we have not lost anything.

Proposition 12.7. Suppose $\mathcal{P}(\mathcal{T})$ is a \mathcal{T} -semiring. If (\mathcal{T}, \cdot) also is a group, then $\tilde{\mathcal{T}}$ is a hyperfield.

Proof. To obtain distributivity, we need to reverse the inequality (12.6), given multiplicative inverses in \mathcal{T} . We follow the argument of [74, Theorem 4.A]. Taking $S = \{a\}$ to be a singleton a , we are given

$$\boxplus(aT) = aa^{-1}(\boxplus(aT)) \subseteq a(\boxplus(a^{-1}aT)) = a \boxplus T.$$

\square

12.2. Major examples of hypergroups and hyperfields.

Let us bring in the major examples of [7]. Although the theory presented above formally passes to $\mathcal{P}(\mathcal{T})$, one gets distributivity in $\mathcal{P}(\mathcal{T})$ precisely when the underlying hyperring satisfies generalized distributivity, which happens in many of the examples. Even better, we will often identify $\mathcal{P}(\mathcal{T})$ with a familiar semiring.

Many hyper-semigroups satisfy the extra property:

Property P. $\{a, b\} \subseteq a \boxplus b$ whenever $a \boxplus b$ is not a singleton.

Note that $a \in a \boxplus a$ iff $a \preceq a^\circ$ in the language of systems, so this has a tropical flavor.

Example 12.8. The first four examples correspond to $(-)$ -bipotent systems, but the last three do not. The complications arise when Example 5.25 is not applicable.

- The tropical hyperfield. Define $\mathbb{R}_\infty = \mathbb{R} \cup \{-\infty\}$ and define the product $a \odot b := a + b$ and

$$a \boxplus b = \begin{cases} \max(a, b) & \text{if } a \neq b, \\ \{c : c \leq a\} & \text{if } a = b. \end{cases}$$

Thus 0 is the multiplicative identity, $-\infty$ is the additive identity, and we have a hyperfield (satisfying Property P), called the **tropical hyperfield**. (It actually is a special case of the example in Lemma 6.59.) This is easily seen to be isomorphic (as hyperfields) to Izhakian's **extended tropical arithmetic** [38], as further expounded as **supertropical algebra** in [47], where we identify $(-\infty, a] := \{c : c \leq a\}$ with a^\vee , so we have a natural hyperfield isomorphism of this tropical hyperfield with the sub-semiring $\widehat{\mathbb{R}_\infty}$ of $\mathcal{P}(\mathbb{R}_\infty)$, because

$$(-\infty, a] + b = \begin{cases} b & \text{if } b > a; \\ (-\infty, a] & \text{if } b = a \\ (-\infty, b] \cup (b, a] = (-\infty, a] & \text{if } b < a. \end{cases}$$

This isomorphism is as semirings. Thus Example 5.25 is applicable.

- *The Krasner hyperfield.* Let $K = \{0; 1\}$ with the usual operations of Boolean algebra, except that now $1 \boxplus 1 = \{0; 1\}$. The Krasner hyperfield satisfies Property P. Again, this generates a sub-semiring of $\mathcal{P}(K)$ having three elements, and is just the supertropical algebra of the monoid K , where we identify $\{0; 1\}$ with 1^ν . Again, Example 5.25 is applicable.
- *Valuative hyperfields* ([7, Example 2.12]) also are isomorphic to the extended semiring in the sense of [47], in the same way.
- *(Hyperfield of signs)* Let $S := \{0, 1, -1\}$ with the usual multiplication law and hyperaddition defined by $1 \boxplus 1 = \{1\}$, $-1 \boxplus -1 = \{-1\}$, $x \boxplus 0 = 0 \boxplus x = \{x\}$, and $1 \boxplus -1 = -1 \boxplus 1 = \{0, 1, -1\} = S$. Then S is a hyperfield (satisfying Property P), called the **hyperfield of signs**.

As already noted in [29, Example 6.9], the four elements $\{\{0\}, \{-1\}, \{1\}, S\}$ constitute the sub-semiring[†] of $\mathcal{P}(S)$, and comprises a meta-tangible system, as noted in Example 6.52.

- *The phase hyperfield.* Let S^1 denote the complex unit circle, and $P := S^1 \cup \{0\}$. Points a and b are **antipodes** if $a = -b$. Multiplication is defined as usual (so corresponds on S^1 to addition of angles). We call an arc of less than 180 degrees **short**. The hypersum is given by

$$a \boxplus b = \begin{cases} \text{all points in the short arc from } a \text{ to } b \text{ if } a \neq b; \\ \{-a, 0, a\} \text{ if } a = -b \neq 0; \\ \{a\} \text{ if } b = 0. \end{cases}$$

Then P is a hyperfield (satisfying Property P), called the **phase hyperfield**. At the power set level, given $T_1, T_2 \subseteq S^1$, one of which having at least two points, we define $T_1 \boxplus T_2$ to be the union of all (short) arcs from a point of T_1 to a non-antipodal point in T_2 (which together makes a connected arc), together with $\{0\}$ if T_2 contains an antipode of T_1 . Thus the system is not meta-tangible. Note that any proper arc of S^1 can be obtained by taking T_1 to be a single point in the middle and T_2 to be the two endpoints. $S^1 \cup \{0\}$ itself is obtained as the sum of two antipodal short arcs which are not points. Thus the system has height ≤ 3 .

In other words, the sub-semiring $\widehat{S^1}$ of $\mathcal{P}(S^1)$ is the set of arcs, possibly with $\{0\}$ adjoined, where \boxplus is concatenation (and filling in the rest of S^1 if the arcs go more than half way around), and adjoining $\{0\}$ if the arcs contain an antipode.

Double distributivity fails, when we take a_1 and a_2 almost to be antipodes, $b_1 = a_2$, and the arc connecting b_1 and b_2 just passes the antipode of a_1 ; then $(a_1 \boxplus a_2)(b_1 \boxplus b_2)$ is the arc from a_1 to b_2 , a little more than a semicircle, whereas $a_1 b_1 \boxplus a_1 b_2 \boxplus a_2 b_2$ is already all of S^1 . It would be interesting to test Conditions A1 and A2 on matrices over this system.

Viro [74] also has a somewhat different version.

- The “triangle” hyperfield A defined over \mathbb{R}^+ by the formula

$$a \boxplus b = \{c \in \mathbb{R}^+ : |a - b| \leq c \leq a + b\}.$$

In other words, $c \in a \boxplus b$ iff there exists an Euclidean triangle with sides of lengths a, b , and c . The triangle hyperfield is not doubly distributive but does satisfy Property P, since $|a - b| \leq a \leq a + b$. Here $\{[a_1, a_2] : a_1 \leq a_2\}$, although not meta-tangible, (and not equal to \mathcal{T}^+) has height 2 since $[a_1, a_2] = \frac{a_1 + a_2}{2} + \frac{a_2 - a_1}{2} \in \hat{A}$ whereas $[a_1, a_2] + [a'_1, a'_2]$ is some interval going up to $a_2 + a'_2$.

- Here is another example, suggested by Lopez, also cf. [31]. Consider \mathbb{R} , with addition given by $a \boxplus b$ and $b \boxplus a$ (for $a \leq b$) to be the interval $[a, b]$. This extends to addition on intervals, given by $[a_1, b_1] + [a_2, b_2] = \{\min(a_1, a_2), \max(b_1, b_2)\}$, which clearly is associative. But the hyperinverse is not unique, since $a + (-a) = [-a, a]$ contains 0, as does $\frac{a}{2} + (-a)$. On the other hand, this does satisfy the restriction that every set of the form $a + (-a)$ cannot be of the form $a + (-b)$ for $b \neq a$, so if we modify the condition of hypernegative to stipulate that $a + (-a)$ must be written in the form $c + (-c)$ for some c , then a is unique.

12.2.1. Summary of the transfer from hyperrings and hypermodules to systems.

Given a hypermodule M over a hyperring \mathcal{T} (where possibly $M = \mathcal{T}$), we want to work in the \mathcal{T} -module $\mathcal{P}(M)$ with negation map. M is identified with the singletons of $\mathcal{P}(M)$, which we called the **tangible** elements; these apply for example to the tropical theory. In order to consider distributivity we need to pass to $\{\boxplus a_i : a_i \in M\}$. Associativity requires passing further to $\{\boxplus S : |S| = 3\}$, which leads us to $\widetilde{\mathcal{T}(M)} := \{\boxplus S : |S| < \infty\}$.

We extend the given a hyper-negation to $(-)$ on $\mathcal{P}(M)$, and define

$$\widetilde{\mathcal{T}(M)}^\circ = \{a \boxplus ((-)a) : a \in \widetilde{\mathcal{T}(M)}\},$$

the \mathcal{T} -submodule of hyperzeros of $\widetilde{\mathcal{T}(M)}$. ($\widetilde{\mathcal{T}}^\circ$ is an ideal of $\widetilde{\mathcal{T}}$.) This will take the place of zero when extending results from classical algebra. The relevant system is $(\widetilde{\mathcal{T}(M)}, \mathcal{T}(M), (-), \preceq)$.

13. APPENDIX B: FUZZY RINGS

A concept similar to systems was introduced in 1986 and refined in 2011 by Dress [23] and Wenzel [24]. This treatment is inspired by [29]. Let \mathcal{A}^\times denote the set of invertible elements of a pre-semiring \mathcal{A} .

Definition 13.1. [24, Definitions 2.1,2.8],[29, Definition 2.14] A **fuzzy ring** is an \mathcal{A}^\times -semiring $(\mathcal{A}, +, \cdot, 0, 1)$ together with a distinguished element ε and a proper \mathcal{A}^\times -semiring ideal \mathcal{A}_0 satisfying the following axioms:

- (i) $\varepsilon^2 = 1$;
- (ii) $a \in \mathcal{A}^\times$, with $1 + a \in \mathcal{A}_0$ iff $a = \varepsilon$.
- (iii) If $a_i \in \mathcal{A}$, with $a_1 + a_2, a_3 + a_4 \in \mathcal{A}_0$, then $a_1 a_3 + \varepsilon a_2 a_4 \in \mathcal{A}_0$.
- (iv) If $a_i \in \mathcal{A}$, with $a_1 + a_2(a_3 + a_4) \in \mathcal{A}_0$, then $a_1 + a_2 a_3 + a_2 a_4 \in \mathcal{A}_0$.

The fuzzy ring is **coherent** if \mathcal{A}^\times spans $(\mathcal{A}, +)$.

In line with our earlier approach, it is natural to generalize the definition slightly, and insert an extra set into the definition, taking \mathcal{T} instead of \mathcal{A}^\times . On the other hand, conditions (iii) and (iv) do not enter into our proofs (and also did not enter into the proof of [29, Theorem 3.3]). This motivates us to delete them.

Definition 13.2. [29, Definition 2.14] A **fuzzy \mathcal{T} -ring** is a \mathcal{T} -cancellative \mathcal{T} -semiring $(\mathcal{A}, +, \cdot, 0, 1)$ where \mathcal{T} is a multiplicative submonoid of \mathcal{A} , together with a distinguished element $\varepsilon \in \mathcal{T}$ and a proper \mathcal{T} -semiring ideal \mathcal{A}_0 satisfying the following axioms:

- (i) $\varepsilon^2 = 1$;
- (ii) For any $a_i \in \mathcal{T}$, $a_1 + a_2 \in \mathcal{A}_0$ iff $a_1 = \varepsilon a_2$.
- (iii) $\mathcal{T}_0 \cap \mathcal{A}_0 = \{0\}$.

The fuzzy ring is **\mathcal{T} -coherent** if \mathcal{T} spans $(\mathcal{A}, +)$.

We are back to fuzzy rings when we take $\mathcal{T} = \mathcal{A}^\times$.

Definition 13.3. A fuzzy \mathcal{T} -ring is **meta-tangible** if $a + b \in \mathcal{T}$ whenever $a, b \in \mathcal{T}$ with $a \neq \varepsilon b$.

Remark 13.4.

- (i) The sub- \mathcal{T} -semiring generated by \mathcal{T} and \mathcal{A}_0 is clearly fuzzy, so we assume from now on that it equals \mathcal{A} .
- (ii) In a meta-tangible fuzzy \mathcal{T} -ring \mathcal{A} , we can also replace \mathcal{A}_0 by $\{a + \varepsilon a : a \in \mathcal{T}\}$, in which case \mathcal{A} becomes coherent.
- (iii) Condition (ii) of Definition 13.2 matches Definition 13.1(ii) for $a_i \in \mathcal{A}^\times$.

Definition 13.5. A fuzzy \mathcal{T} -ring is **matroidal** if

$$a + b \in \mathcal{A}_0 \quad \text{implies} \quad b = \varepsilon a + c \text{ for some } c \in \mathcal{A}_0. \quad (13.1)$$

The next result reconciles Definitions 13.1 and 13.2.

Lemma 13.6. (Assuming $\mathcal{A} = \mathcal{T} + \mathcal{A}_0$)

- (i) Condition (iii) holds whenever $a_1, a_2 \in \mathcal{T}$.
- (ii) Condition (iii) holds for any matroidal fuzzy \mathcal{T} -ring.

Proof. (i) $a_1 + a_2 \in \mathcal{A}_0$, so $a_1 = \varepsilon a_2$ by Definition 13.2(ii). Hence $a_1 a_3 + \varepsilon a_2 a_4 = a_1(a_3 + a_4) \in \mathcal{A}_0$.

(ii) $a_1 + a_2 \in \mathcal{A}_0$, so, by Definition 13.3, $a_1 + b = \varepsilon a_2$ for some $b \in \mathcal{A}_0$. Hence $a_1 a_3 + \varepsilon a_2 a_4 = a_1(a_3 + a_4) + b a_4 \in \mathcal{A}_0$. \square

Proposition 13.7. *Any fuzzy \mathcal{T} -ring gives rise to a uniquely negated triple $(\mathcal{A}, \mathcal{T}, (-))$, where $(-)a = \varepsilon a$. Furthermore, $\mathcal{A}^\circ \subseteq \mathcal{A}_0$.*

Proof. The map $a \mapsto \varepsilon a$ obviously is a negation map. Furthermore, if $a(-)b \in \mathcal{A}^\circ$ for $a, b \in \mathcal{T}$, then $1 + \varepsilon ba^{-1} \in \mathcal{A}^\circ$, implying $\varepsilon ba^{-1} = \varepsilon$, and thus $b = a$, proving the triple is uniquely negated.

$$a(-)a = a + \varepsilon a \in \mathcal{A}_0. \quad \square$$

If one wants to strengthen the link from the theory of systems to fuzzy rings, one defines \preceq by saying $a \preceq_\circ b$ when $a = b + c$ for some $c \in \mathcal{A}_0$. This yields Condition (iv) of Definition 13.2 and incorporates \mathcal{A}_0 into the formal definition of system (as an ideal containing \mathcal{A}° disjoint from \mathcal{T}).

13.0.2. Fuzzy rings versus uniquely negated triples.

Let us link fuzzy rings to uniquely negated triples.

Conversely to Proposition 13.7, the notion of fuzzy \mathcal{T} -ring encompasses uniquely negated systems.

Proposition 13.8. *Suppose that $\mathcal{S} := (\mathcal{A}, \mathcal{T}, (-), \preceq_\circ)$ is a cancellative, uniquely negated system, where \mathcal{A} is a \mathcal{T} -semiring. Then \mathcal{S} gives rise to a fuzzy \mathcal{T} -ring \mathcal{A}' with the same operations, where $\mathcal{A}_0 = \mathcal{A}^\circ$ and $(\mathcal{A}', +)$ is generated by \mathcal{T} and \mathcal{A}_0 , and $\varepsilon = (-)1$.*

Proof. Note that $(\mathcal{A}', \mathcal{T}, (-), \preceq_\circ)$ also is a system, so we may assume that $\mathcal{A}' = \mathcal{A}$. Properties (i) and (ii) of Definition 13.2 are clear, and (iii) is by Corollary 5.3. \square

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